Termes du 2ème ordre pour l'interprétation du potentiel

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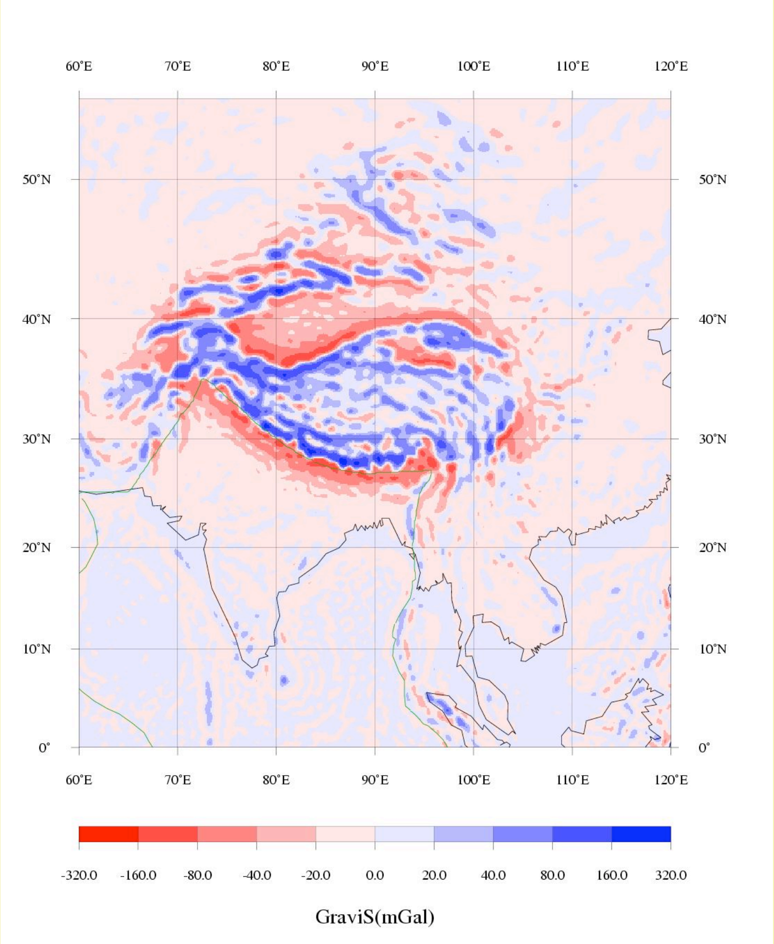
«A first requirement of a reference Earth model is that it must fit the mean radius, mass and inertia. » Bullen (1974)



Δg_{AL}, 2ème ordre

Potentiel de gravitation externe :

$$\varphi(r,\theta,\lambda) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{b}{r}\right)^{\ell+1} \varphi_\ell^m(b)\, Y_\ell^m(\theta,\lambda).$$

avec :

$$\varphi_\ell^m(b) = -\frac{G}{(2\ell+1)b^{\ell+1}} \int_V \rho(r,\theta,\lambda) r^\ell Y_\ell^m(\theta,\lambda)\, \mathrm{dV}$$

$$\boxed{\phi_\ell^m = \int_V \rho(r,\theta,\lambda) r^\ell Y_\ell^m(\theta,\lambda)\, \mathrm{d}V.}$$

($\phi_0^0 = \mathcal{M}$ ϕ_1^m : centre de masse

ϕ_2^m : tenseur d'inertie)

Terre = modèle sphérique + variations latérales

$$\rho(r, \theta,\lambda) = \rho_0(r) + \delta\rho\,(r, \theta,\lambda)$$

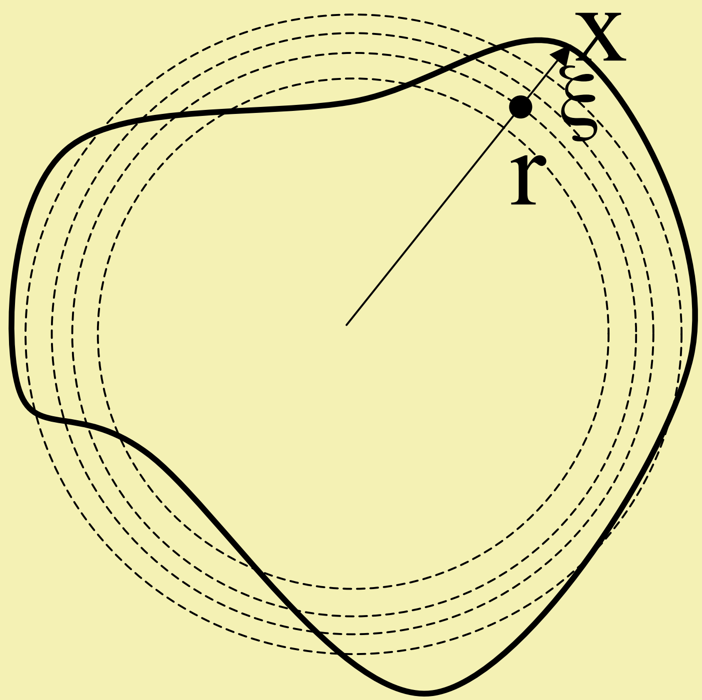
$$\phi\,(\rho) = \phi\,(\rho_0) + \delta\phi\,(\delta\rho,\xi)$$

I. Définition de ρ_0 , signification de $\delta\rho$?

II. Le calcul de $\delta\phi$ au 1er ordre suffit-il ? (isostasie)

III. Déterminations de M, I, b, I/M => I_0/M_0.

I. Le modèle moyen $\rho_0\,(r)$.



$\rho_0\,(r) = < \rho\,(r) >$ ou $\rho_0\,(r) = < \rho\,(x) >$.

↓ Perte des discontinuités
Variations latérales importantes

↓ Commencer par définir le
rayon moyen r

Rayon moyen et densité moyenne

$$r = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} ||x(\theta, \lambda)|| \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\lambda$$

$$r = \langle \| x \| \rangle = \| x \| \big|_0 \quad \Rightarrow \quad \xi \big|_0 = (\| x \| - r) \big|_0 = 0$$

$$\rho_0(r) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \rho(x(r, \theta, \lambda)) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\lambda.$$

$$\rho_0(r) = \langle \rho(x) \rangle = \rho(x) \big|_0$$

II. La masse et l'inertie

$$\mathcal{M} = \int_V \rho \, \mathrm{d}V.$$

$$\mathcal{I} = \frac{1}{3}\mathrm{tr}\mathcal{J} = \frac{2}{3}\int_V \rho x^2 \, \mathrm{d}V$$

$$\mathcal{M}_0 = \int_{V_0} \rho_0 \, \mathrm{d}V = 4\pi \int_0^b \rho_0(r) r^2 \, \mathrm{d}r,$$

$$\mathcal{I}_0 = \frac{2}{3}\int_{V_0} \rho_0 r^2 \, \mathrm{d}V = \frac{8\pi}{3}\int_0^b \rho_0(r) r^4 \, \mathrm{d}r.$$

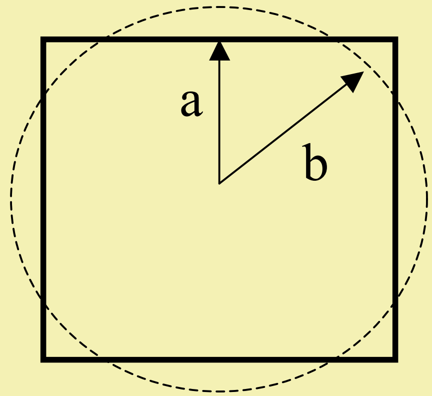
$$\delta\mathcal{I} = \mathcal{I} - \mathcal{I}_0$$

$$\delta\mathcal{M} = \mathcal{M} - \mathcal{M}_0$$

?

Plus généralement : $\delta\phi = \phi(\rho) - \phi(\rho_0)$?

Exemple :



Cube homogène de côté 2a, densité ρ :

$$\mathcal{M} = 8\rho a^3$$

$$\mathcal{I} = \frac{1}{3}\rho a^5$$

Sphère homogène de rayon b, densité ρ_0

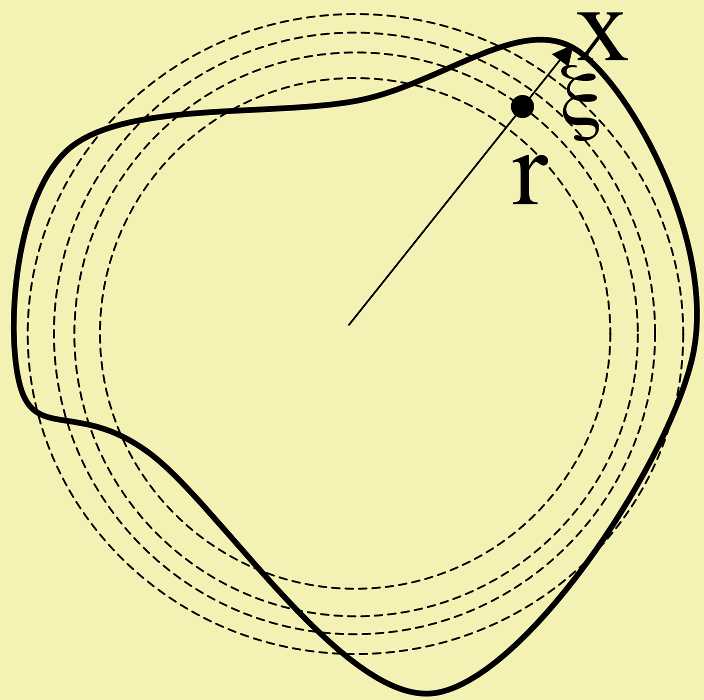
$$\mathcal{M}_0 = \frac{4\pi}{3}\rho_0 b^3$$

$$\mathcal{I}_0 = \frac{8\pi}{15}\rho_0 b^5$$

On ne peut pas conserver ρ, M et I.

…comme en cartographie…

Perturbation de la masse



$$\mathcal{M} = \int_V \rho \, \mathrm{d}V.$$

$\delta_l \rho = \rho(x) - \rho_0(r)$

$\xi = x - r$

Au 1er ordre :

$$\delta_1 \mathcal{M} = \int_V (\delta_\ell \rho + \rho \mathrm{div} \xi) \, \mathrm{d}V \qquad = 0$$

Au 2ème ordre :

$$\delta \mathcal{M} = \delta_1 \mathcal{M} + \frac{1}{2} \delta_2 \mathcal{M} = \frac{1}{2} \delta_2 \mathcal{M},$$

$$= \int_V \left\{ \delta_\ell \rho \, \mathrm{div} \xi + \rho \, \mathrm{div} \left(\frac{\xi^2}{r} e_r \right) \right\} \mathrm{d}V \qquad \text{non nul}$$

$$\xi_r = \xi_h + \xi_d, \qquad \delta \mathcal{M} = \delta_h \mathcal{M} + \delta_d \mathcal{M},$$

Composante hydrostatique $(\boldsymbol{\xi} = \xi_h \boldsymbol{e}_r \quad \delta_l \rho = 0)$

$$\delta_h \mathcal{M} = \int_V \rho \frac{\xi_h}{r} \left(2\partial_r \xi_h + \frac{\xi_h}{r} \right) \, \mathrm{d}V$$

$$\xi_h(r,\theta,\lambda) = -\frac{2}{3\sqrt{5}} r\epsilon(r) Y_2^0(\theta,\lambda),$$

$$\frac{\delta_h \mathcal{M}}{\mathcal{M}} \simeq 2.7 \times 10^{-6} \qquad \frac{\delta_h \mathcal{I}}{\mathcal{I}} \simeq 9.4 \times 10^{-6}.$$

$$\frac{\delta_h(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}} = \frac{\delta_h \mathcal{I}}{\mathcal{I}} - \frac{\delta_h \mathcal{M}}{\mathcal{M}} \simeq 6.7 \times 10^{-6}.$$

Composante non hydrostatique

$$\delta_d \mathcal{M} = \int_V \delta_\ell \rho \, \mathrm{div}(\xi_h e_r) \, \mathrm{d}V + \int_V \delta_\ell \rho \, \mathrm{div}(\xi_d e_r) \, \mathrm{d}V - \int_V \partial_r \rho \frac{\xi_d^2}{r} \, \mathrm{d}V - \sum_{r_\Sigma} \int_\Sigma [\rho] \frac{\xi_d^2}{r_\Sigma} \, \mathrm{d}\Sigma.$$

$$\delta_\Sigma \mathcal{M} = -4\pi \sum_{r=r_\Sigma} r_\Sigma \, [\rho] \, \left(\xi_d^2\right)\Big|_0 .$$

$$\left| \frac{\delta_d \mathcal{M}}{\mathcal{M}} \right| \leq 4 \frac{\delta_\Sigma \mathcal{M}}{\mathcal{M}} \leq 4.4 \times 10^{-6}.$$

$$\left| \frac{\delta_d \mathcal{I}}{\mathcal{I}} \right| \leq 16.4 \times 10^{-6}.$$

$$\left| \frac{\delta_d (\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}} \right| \leq 2 \left(\left| \frac{\delta_\Sigma \mathcal{I}}{\mathcal{I}} - \frac{\delta_\Sigma \mathcal{M}}{\mathcal{M}} \right| + \frac{\delta_\Sigma \mathcal{I}}{\mathcal{I}} + \frac{\delta_\Sigma \mathcal{M}}{\mathcal{M}} \right)$$

$$= 4 \max \left(\frac{\delta_\Sigma \mathcal{I}}{\mathcal{I}}, \frac{\delta_\Sigma \mathcal{M}}{\mathcal{M}} \right) \leq 1.64 \times 10^{-5}.$$

On retranche la composante hydrostatique :

$$\frac{\mathcal{I}_0}{\mathcal{M}_0} = \frac{\mathcal{I}}{\mathcal{M}}\left(1 - \frac{\delta_h(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}}\right)$$

$$= (1.342\,332 \pm 0.000\,031) \times 10^{13}\mathrm{m}^2.$$

On ajoute le majorant de la composante non hydrostatique à l'erreur :

$$\frac{\mathrm{d}(\mathcal{I}_0/\mathcal{M}_0)}{\mathcal{I}_0/\mathcal{M}_0} = \frac{\mathrm{d}(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}} + \left|\frac{\delta_d(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}}\right| \leq 2.3 \times 10^{-5},$$

Observables

Le rayon

$$b = R + h|_0,$$

$$R = (6\,370\,994.4 \pm 3)\ \mathrm{m} \qquad \left(\frac{\mathrm{d}R}{R} = 4.7 \times 10^{-7}\right).$$

$$b = (6\,371\,230 \pm 10)\ \mathrm{m} \qquad \left(\frac{\mathrm{d}b}{b} = 1.6 \times 10^{-6}\right).$$

Model	Angular resolution	Years of development	$h\|_0$ (m)
FNOC	10' $\times$ 10'	1960's $\rightarrow$ 1984	237.2
ETOPO5	5' $\times$ 5'	1980's	233.1
smoothed ETOPO5	30' $\times$ 30'		231.4
TerrainBase	5' $\times$ 5'	1994	234.3
DTM5	5' $\times$ 5'	$\rightarrow$ 1995	230.7
JGP95E	5' $\times$ 5'	$\rightarrow$ 1994-1995	231.4

Mean land elevation (m)	Continental area (%)	Year of publication	Infered $h\|_0$ (m)
771 (a)	30 (b)	1921	231
875 (a)	29.2 (a)	1933	255.5
801 (a)	30 (b)	(c)	240
756 (d)	29.1 (d)	1967	220
726	30.3 (e)	1973	220 (e)

La masse

$$G\mathcal{M} = (398\,600\,441.5 \pm 4.0) \times 10^{6}\ \mathrm{m}^{3}\ \mathrm{s}^{-2} \qquad \left(\frac{\mathrm{d}(G\mathcal{M})}{G\mathcal{M}} = 10^{-8} \right).$$

$$G = (6.673 \pm 0.010) \times 10^{-11}\ \mathrm{m}^{3}\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2} \qquad \left(\frac{\mathrm{d}G}{G} = 1.5\ \times 10^{-3} \right).$$

(Mohr & Taylor, 1999)

$$\mathcal{M} = (5.9733 \pm 0.0090)\ \times 10^{24}\ \mathrm{kg} \qquad \left(\frac{\mathrm{d}\mathcal{M}}{\mathcal{M}} = 1.5\ \times 10^{-3} \right).$$

Reference [and name of the gravitational model].	$G\mathcal{M}$ and uncertainty		
	(10^6 m^3 /s ^{2})		
Lerch et al. 1978	398 600	440	20
Smith et al. 1985 (a)		434	2
Marsh et al. 1985 (a)		434	5
Tapley et al. 1985 (a)		440	2
Newhall et al. 1987 (a)		443	6
Ries et al. 1989 (a)		440.5	1
Marsh et al. [GEM-T2] 1989 (b)		436	
Rapp et al. [OSU91] 1991 (c)		440	
Ries et al. 1992		441.5	0.8
Schwintzer et al. [GRIM4] 1997		437.7	0.2
Lemoine et al. [EGM96] 1998 (d)		443.2	0.4

L'inertie

$$\mathcal{J}_{11} = \mathcal{I} + \frac{\sqrt{5}}{3G}\left(C_{20} - \sqrt{3}C_{22}\right) \qquad \mathcal{J}_{22} = \mathcal{I} + \frac{\sqrt{5}}{3G}\left(C_{20} + \sqrt{3}C_{22}\right)$$

$$\mathcal{J}_{33} = \mathcal{I} - 2\frac{\sqrt{5}}{3G}C_{20} \qquad \mathcal{J}_{12} = -\sqrt{\frac{5}{3}}\frac{S_{22}}{G}$$

$$\mathcal{J}_{13} = -\sqrt{\frac{5}{3}}\frac{C_{21}}{G} \qquad \mathcal{J}_{23} = -\sqrt{\frac{5}{3}}\frac{S_{21}}{G}.$$

$$\mathcal{H} = \frac{C - \frac{1}{2}(A+B)}{C},$$

$$J_2 = \frac{C - \frac{1}{2}(A+B)}{\mathcal{M}a_e^{*2}}$$

$$\rightarrow \frac{C}{\mathcal{M}a_e^{*2}} = \frac{J_2}{\mathcal{H}}. \rightarrow \frac{\mathcal{I}}{\mathcal{M}b^2} = \frac{J_2}{\mathcal{H}}\left(1 - \frac{2}{3}\mathcal{H}\right)\left(\frac{a_e^*}{b}\right)^2.$$

$$\mathcal{H} = (327\,379 \pm 2) \times 10^{-8} \qquad \left(\frac{\mathrm{d}\mathcal{H}}{\mathcal{H}} = 6.1 \times 10^{-6} \right).$$ (Dehant & Capitaine 1996)

$$J^n_{2\,(2000)} = J_{2\,(2000-\Delta t)} \left(\frac{a_e^*}{a_e} \right)^2 + \partial_t J_2\, \Delta t.$$ (tide free)

Model	a_e^* (m)	Reference year	J_2	$J^n_{2\,(2000)}$
			in units of 10^{-3}	
OSU91	6 378 137	1986 (a)	1.082 627 04	1.082 626 68
JGM2	6 378 136.3	1986	6 93	6 33
GRIM4	6 378 136	1984	7 19 (22)	6 43
EGM96	6 378 136.3	1986	6 68 (8)	6 08

$$J^n_{2\,(2000)} = (1\,082\,626.4 \pm 0.5) \times 10^{-9} \qquad \left(\frac{\mathrm{d}J^n_2}{J^n_2} = 4.6 \times 10^{-7} \right).$$

$$J^m_2 = J^z_2 + \Delta J_2, \quad J^z_2 = J^n_2 + k_2 \Delta J_2,$$

(mean tide, zero frequency tide)

$$J^z_2 = (1\,082\,617.1 \pm 0.5) \times 10^{-9} \qquad \left(\frac{\mathrm{d}J^z_2}{J^z_2} = 4.6 \times 10^{-7} \right).$$

(exclu les marées directes)

$$\frac{\mathcal{I}}{\mathcal{M}} = (1.342\,341 \pm 0.000\,009) \times 10^{13} \mathrm{m}^2 \quad \left(\frac{\mathrm{d}(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}} = 6.6 \times 10^{-6} \right),$$

Coefficients d'inertie

$$\frac{\mathcal{I}_0}{\mathcal{M}_0 b^2} = 0.330\,684 \pm 9 \times 10^{-6}$$

$$\left(\frac{\mathrm{d}(\mathcal{I}_0/\mathcal{M}_0 b^2)}{\mathcal{I}_0/\mathcal{M}_0 b^2} = 2.6 \times 10^{-5} \right).$$

$$\frac{\mathcal{I}}{\mathcal{M} b^2} = 0.330\,692 \pm 3 \times 10^{-6}$$

$$\left(\frac{\mathrm{d}(\mathcal{I}/\mathcal{M} b^2)}{\mathcal{I}/\mathcal{M} b^2} = 0.97 \times 10^{-5} \right).$$

$$\frac{\mathcal{I}_0}{\mathcal{M}_0 R^2} = 0.330\,708 \pm 8 \times 10^{-6}$$

Romanowicz & Lambeck (1977) :

$$\frac{\mathcal{I}}{\mathcal{M} R^2} = 0.330\,850 \pm 15 \times 10^{-6}$$

Yoder (Handbook, 1995) :

$$\frac{\mathcal{I}}{\mathcal{M} R^2} = 0.330\,714\,4 \pm ? \times 10^{-6}$$

Variations temporelles

$$\frac{\partial_t(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}} = \frac{\partial_t\mathcal{I}}{\mathcal{I}} - \frac{\partial_t\mathcal{M}}{\mathcal{M}} = \frac{\partial_t J_2}{J_2} - \frac{\partial_t\mathcal{H}}{\mathcal{H}}\frac{1}{1-\frac{2}{3}\mathcal{H}}$$

$$\partial_t\mathcal{I} = \frac{2}{3}\int_V \rho\,\frac{\mathrm{d}x^2}{\mathrm{d}t}\,\mathrm{d}V = \frac{4}{3}\int_V v\cdot x\,\mathrm{d}m,$$

$$\frac{\partial_t(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}} = \frac{\partial_t\mathcal{I}}{\mathcal{I}} = \frac{4}{3b}\frac{\mathcal{M}b^2}{\mathcal{I}}\frac{1}{\mathcal{M}b}\int_V v\cdot x\,\mathrm{d}m.$$

$|\frac{1}{\mathcal{M}b}\int_V v\cdot x\,\mathrm{d}m| \leq 1$ cm/an =>

$|\partial_t\mathcal{I}/\mathcal{I}| \leq 6.3\times 10^{-9}$ /an $< 9 \quad \partial_t J_2/J_2$

$$\partial_t\mathcal{H}/\mathcal{H} \simeq \partial_t J_2/J_2$$

$$b = (6\,371\,230 \pm 10) \quad \text{m}$$

$$\mathcal{M}_0 = (5.9733 \pm 0.0090) \times 10^{24} \quad \text{kg}$$

$$\mathcal{I}_0/\mathcal{M}_0 = (1.342\,332 \pm 0.000\,031) \times 10^{13} \quad \text{m}^2$$

data	b	$\mathcal{M}$	$\mathcal{I}$	$\mathcal{I}/\mathcal{M}b^2$	$\mathcal{I}/\mathcal{M}$
final uncertainty	1.6×10^{-6}	1.5×10^{-3}	1.5×10^{-3}	2.6×10^{-5}	2.3×10^{-5}
measurement error	1.6×10^{-6}	1.5×10^{-3}	1.5×10^{-3}	9.7×10^{-6}	6.6×10^{-6}
non-hydrostatic error	0	4.4×10^{-6}	16.4×10^{-6}	16.4×10^{-6}	16.4×10^{-6}
ellipticity correction	0	2.7×10^{-6}	9.4×10^{-6}	6.7×10^{-6}	6.7×10^{-6}
Influence of $\delta b = 230$ m	3.6×10^{-5}	0	0	7.2×10^{-5}	0

Data	Symbol	Value (uncertainty)	Unit	Relative uncertainty
Equatorial radius	a_e	6 378 137 (3)	m	4.7×10^{-7}
Geoc. Gravit. constant	$G\mathcal{M}$	3.986 004 415 (40)	10^{14} m^3 s^{-2}	1.0×10^{-8}
Gravitational constant	G	6.673 (10)	10^{-11} m^3 kg^{-1} s^{-2}	1.5×10^{-3}
Mass	$\mathcal{M}$	5.973 3 (90)	10^{24} kg	1.5×10^{-3}
Angular velocity	Ω	7.292 115 (0.1)	10^{-5} rad s^{-1}	1.4×10^{-8}
Tide-free dyn. form factor	J_2^n	1.082 626 4 (5)	10^{-3}	4.6×10^{-7}
Zero-freq dyn. form factor	J_2^z	1.082 617 1(5)	10^{-3}	4.6×10^{-7}
Precession constant	$\mathcal{H}$	3.273 79 (2)	10^{-3}	6.1×10^{-6}
Polar inertia coefficient (a)	$C/\mathcal{M}a_e^2$	0.330 692 (2)		6.6×10^{-6}
2nd equatorial inertia coeff	$B/\mathcal{M}a_e^2$	0.329 613 (2)		6.6×10^{-6}
1st equatorial inertia coeff.	$A/\mathcal{M}a_e^2$	0.329 606 (2)		6.6×10^{-6}
Inertia coefficient (c)	$\mathcal{I}/\mathcal{M}a_e^2$	0.329 971 (2)		6.6×10^{-6}
Inertia coefficient	$\mathcal{I}/\mathcal{M}b^2$	0.330 686 (3)		9.7×10^{-6}
Mean inertia ratio	$\mathcal{I}/\mathcal{M}$	1.342 340 (9)	10^{13} m^2	6.6×10^{-6}
Mean inertia	$\mathcal{I}$	8.018 (12)	10^{37} m^2 kg	1.5×10^{-3}

Data	Symbol	Value (uncertainty)	Unit	Relative uncertainty
Mean solid topography	$h\|_0$	233 (7)	m	3.0×10^{-2}
Mean geoidal radius	R	6 370 994.4 (3.0)	m	4.7×10^{-7}
Physical radius	b	6 371 230 (10)	m	1.6×10^{-6}
Mass	$\mathcal{M}_0$	5.973 3 (90)	10^{24} kg	1.5×10^{-3}
Inertia	$\mathcal{I}_0$	8.018 (12)	10^{37} m^2 kg	1.5×10^{-3}
Inertia ratio (c)	$\mathcal{I}_0/\mathcal{M}_0$	1.342 331 (31)	10^{13} m^2	2.3×10^{-5}
Inertia coefficient (c)	$\mathcal{I}_0/\mathcal{M}_0 b^2$	0.330 684 (9)		2.6×10^{-5}
Inertia coefficient (c)	$\mathcal{I}_0/\mathcal{M}_0 R^2$	0.330 708 (8)		2.3×10^{-5}
2nd radial density moment (d)	ρ_2	5514 (8)	kg m^{-3}	1.5×10^{-3}
4th radial density moment (e)	ρ_4	4558 (7)	kg m^{-3}	1.5×10^{-3}

III. Le potentiel

$$\phi_\ell^m = \int_V \rho(r,\theta,\lambda) r^\ell Y_\ell^m(\theta,\lambda)\,\mathrm{d}V.$$

Au premier ordre :

$$\begin{aligned}\delta_1\phi_\ell^m &= \int_{V_0} \delta_{1e}\rho\, r^\ell Y_\ell^m\,\mathrm{d}V - \int_{\Sigma_0} [\rho]\xi_1^r r^\ell Y_\ell^m\,\mathrm{d}\Sigma \\ &= \int_{V_0} (r^\ell \delta_{1l}\rho + \rho\ \mathrm{div}(\boldsymbol{\xi}_1 r^\ell)) Y_\ell^m\,\mathrm{d}V, \\ &= 4\pi \int_0^b \delta_{1e}\rho_\ell^m\, r^{\ell+2}\,\mathrm{d}r - 4\pi \sum_{r_\Sigma} [\rho]\xi_{1\,\ell}^{r\,m} r^{\ell+2},\end{aligned}$$

$\simeq 0$ (barré)

Au deuxième ordre :

$$\delta\phi_\ell^m = \int_{V_0} \{r^\ell \delta_l \rho + \rho\, \mathrm{div}(\boldsymbol{\xi} r^\ell)\} Y_\ell^m \, \mathrm{d}V$$

$$+ \quad \int_{V_0} \left\{\delta_l \rho\, \mathrm{div}(\boldsymbol{\xi} r^\ell) + (\ell+2)\rho\, \mathrm{div}(r^{\ell-1}(\xi^r)^2\, \boldsymbol{e}_r)/2\right\} Y_\ell^m \, \mathrm{d}V,$$

$$\delta\phi_\ell^m = \delta_h \phi_\ell^m + \delta_d \phi_\ell^m .$$

Hydrostatique Déviatorique

$$\boxed{\delta_d \phi_\ell^m} = L_\ell^m + \boxed{A_\ell^m + B_\ell^m + C_\ell^m + D_\ell^m} = L_\ell^m + \boxed{N_\ell^m},$$

linéaire non linéaire

$$L_\ell^m = \int_{V_0} r^\ell \delta_d \rho Y_\ell^m \,\mathrm{d}V - \int_{\Sigma_0} [\rho] \xi_d r^\ell Y_\ell^m \,\mathrm{d}\Sigma,$$

$$A_\ell^m = -(\ell+2)/2 \int_{\Sigma_0} [\rho] \xi_d^2 r^{\ell-1} Y_\ell^m \,\mathrm{d}\Sigma,$$

$$B_\ell^m = -\int_{\Sigma_0} [\delta_d \rho] \xi_d r^\ell Y_\ell^m \,\mathrm{d}\Sigma,$$

$$C_\ell^m = \int_{V_0} \delta_d \rho \,\mathrm{div}\,(\xi_h r^\ell \boldsymbol{e}_r) Y_\ell^m \,\mathrm{d}V - (\ell+3) \int_{\Sigma_0} [\rho] \xi_d \xi_h r^{\ell-1} Y_\ell^m \,\mathrm{d}\Sigma,$$

$$D_\ell^m = \int_{\Sigma_0} [\rho] \xi_d \xi_h r^{\ell-1} Y_\ell^m \,\mathrm{d}\Sigma.$$

(Avec $\xi_d \to 0$ dans $V_0 \backslash \Sigma_0$)

$$A_\ell^m = -(\ell+2)/2 \int_{\Sigma_0} [\rho]\xi_d^2 r^{\ell-1} Y_\ell^m \,\mathrm{d}\Sigma, \qquad \xi_d^{\mathrm{Moho}} = -\frac{\rho_c}{\rho_m - \rho_c}\xi^{\mathrm{eq}},$$

$$B_\ell^m = -\int_{\Sigma_0} [\delta_d \rho]\xi_d r^\ell Y_\ell^m \,\mathrm{d}\Sigma, \qquad \delta_d\rho = \Delta\rho(O_c - (O_c)_0^0).$$

$$C_\ell^m = \int_{V_0} \delta_d\rho \,\mathrm{div}\,(\xi_h r^\ell \boldsymbol{e}_r) Y_\ell^m \,\mathrm{d}V - (\ell+3)\int_{\Sigma_0} [\rho]\xi_d \xi_h r^{\ell-1} Y_\ell^m \,\mathrm{d}\Sigma,$$

$$\xi_h(r,\theta,\lambda) = -\frac{2}{3\sqrt{5}} r\epsilon(r) Y_2^0(\theta,\lambda),$$

$$C_\ell^m = -\frac{2}{3\sqrt{5}}\epsilon(b)(\ell+3)\left(\int_{V_0} \delta_d\rho\, r^\ell Y_2^0 Y_\ell^m \,\mathrm{d}V - \int_{\Sigma_0} [\rho]\xi_d r^\ell Y_2^0 Y_\ell^m \,\mathrm{d}\Sigma\right)$$

$$\zeta_{C_\ell^m} = -\frac{\epsilon(b)}{6\pi\sqrt{5}}\frac{\ell+3}{2\ell+1} b \int_\Omega \left(2\frac{\delta_d g}{g} + 3\frac{\zeta_d}{b}\right) Y_2^0 Y_\ell^m \,\mathrm{d}\Omega$$

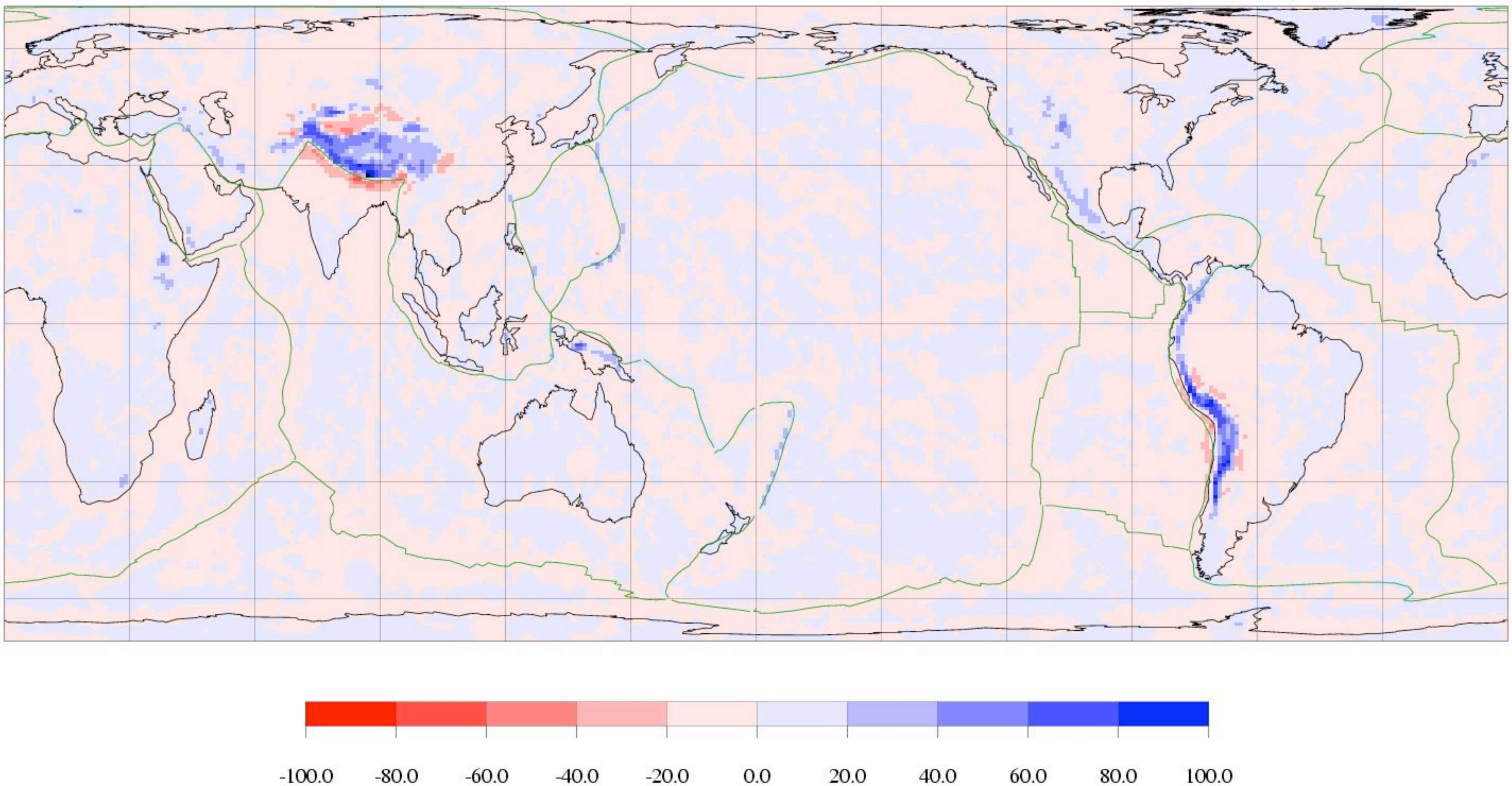
$$D_\ell^m = \int_{\Sigma_0} [\rho]\xi_d\xi_h r^{\ell-1} Y_\ell^m \, \mathrm{d}\Sigma. \qquad \longleftarrow \qquad \sum\nolimits_{r_\Sigma} [\rho]\xi_d = -\rho_m \xi^{\mathrm{eq}} O_c$$

Potentiel, hauteur du géoïde, anomalies à l'air libre

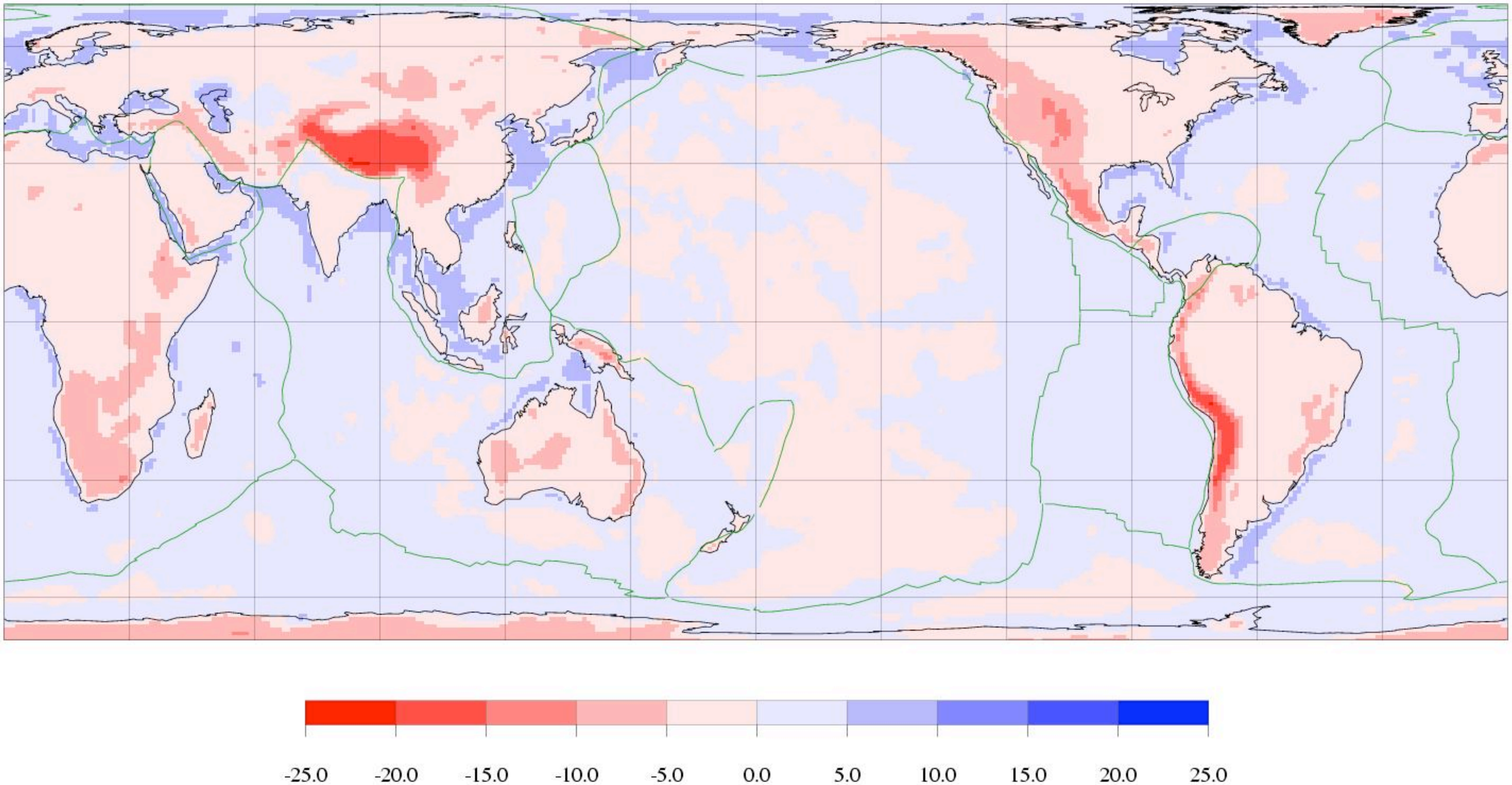
$$\varphi(r, \theta, \lambda) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{b}{r}\right)^{\ell+1} \varphi_\ell^m(b)\, Y_\ell^m(\theta, \lambda).$$

$$\zeta_\ell^m = \frac{\varphi_\ell^m(b)}{g}, \qquad \frac{\delta g_\ell^m}{g} = (\ell - 1)\frac{\zeta_\ell^m}{b}.$$

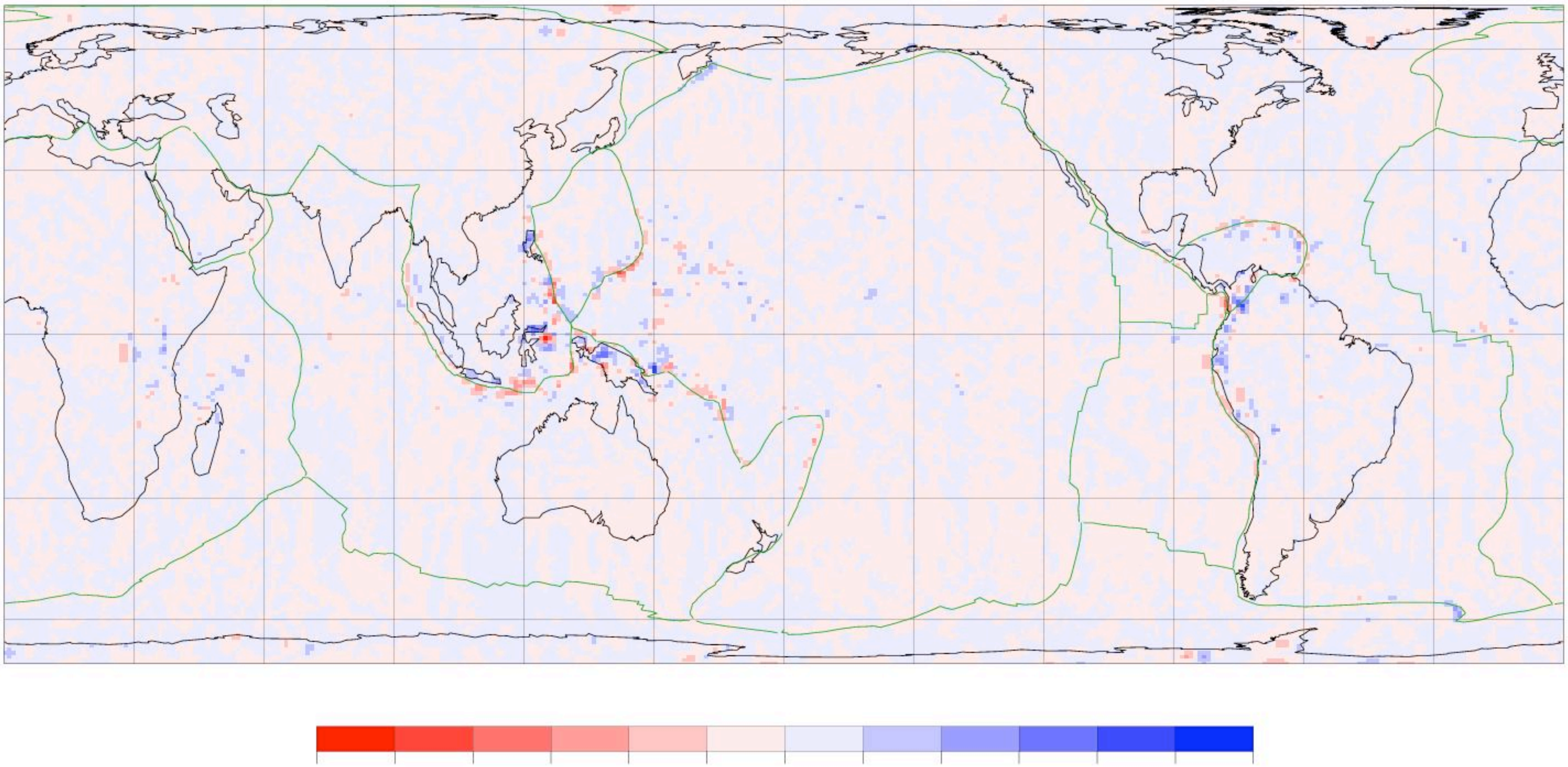
$$\delta g(\theta, \lambda) = -\left(\frac{\partial}{\partial r} + \frac{2}{r}\right) \varphi\,(r = b, \theta, \lambda),$$



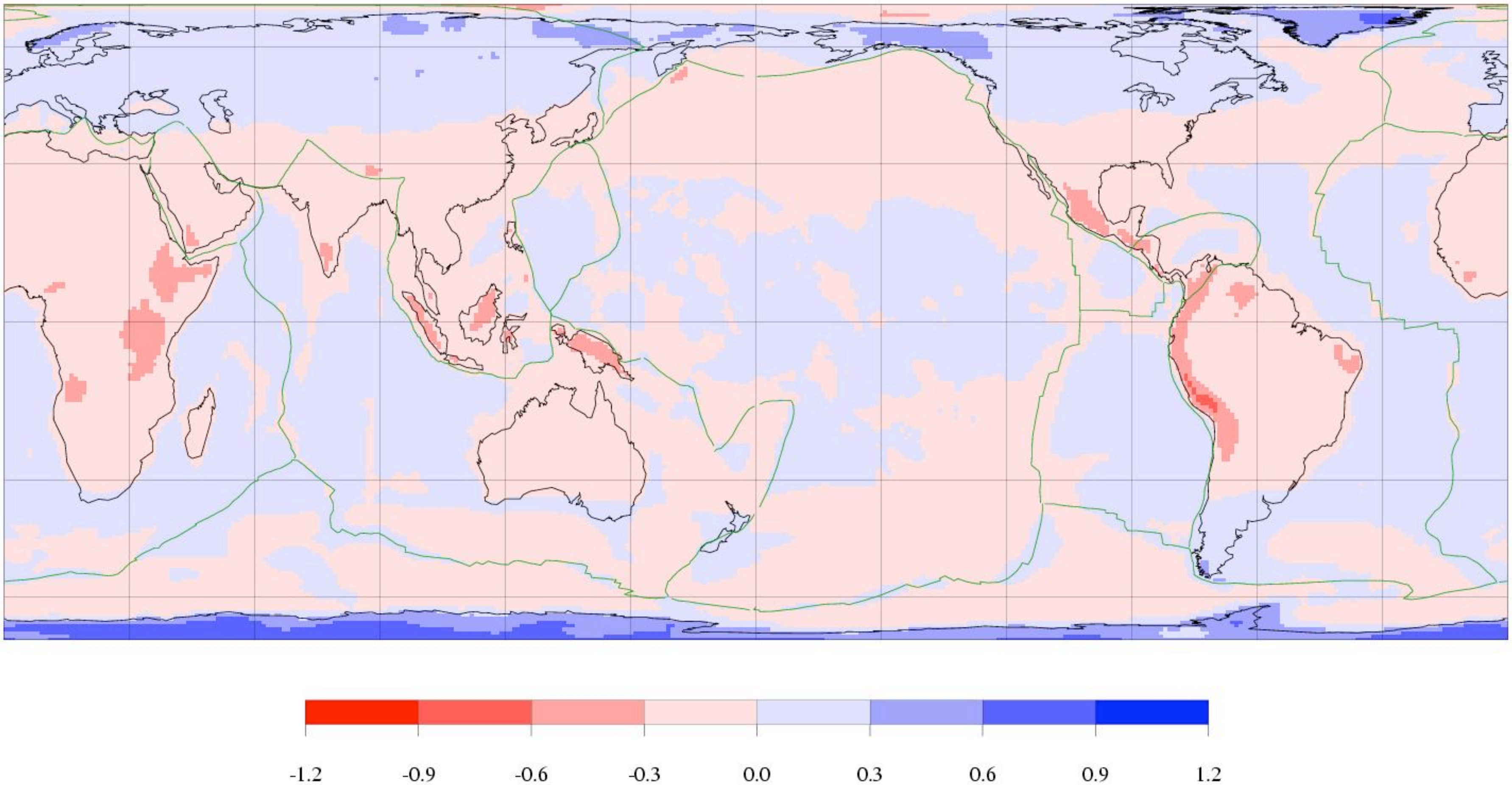
Gravity anomaly (mGal) corresponding to A



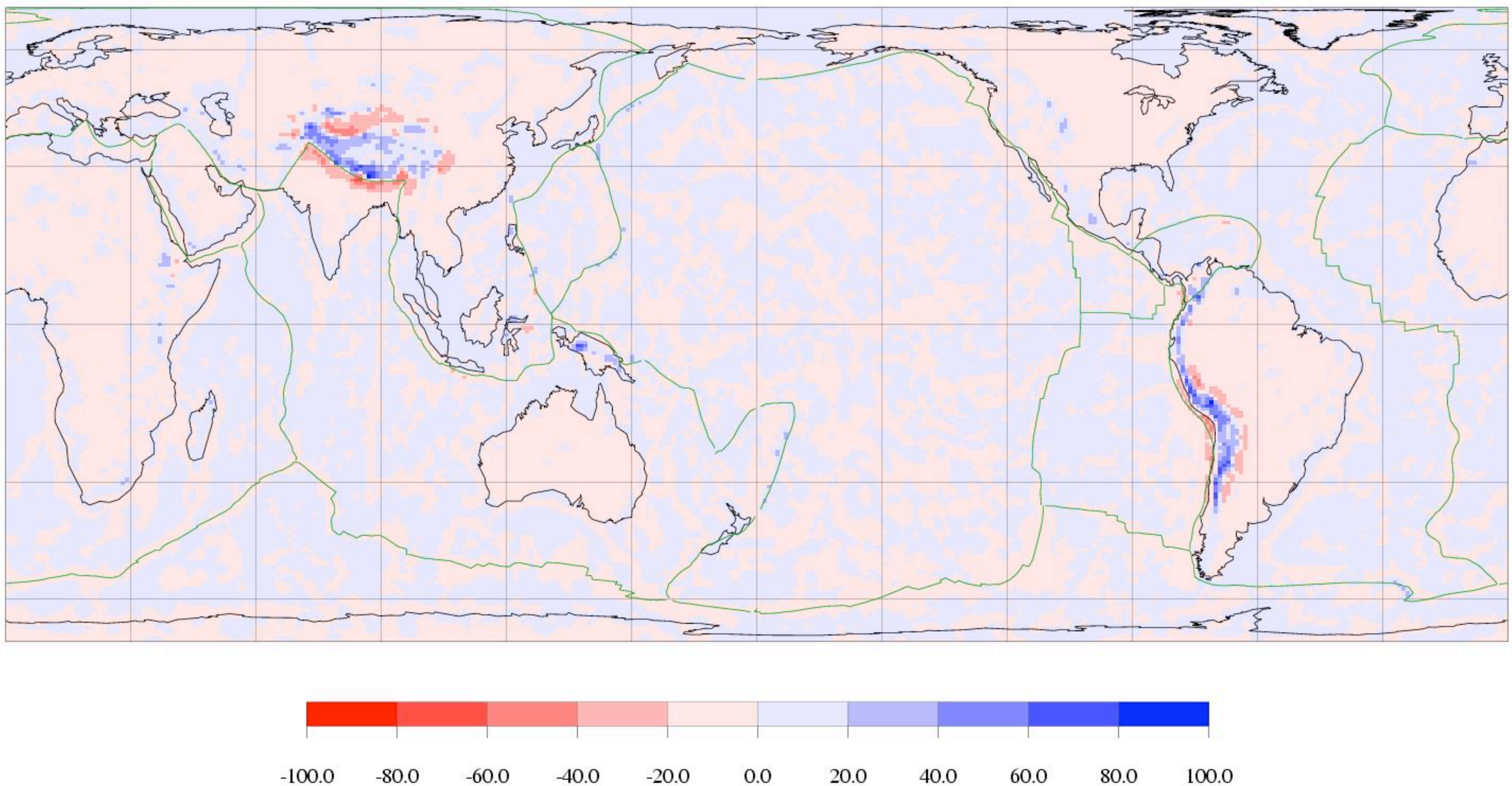
Gravity anomaly (mGal) corresponding to B



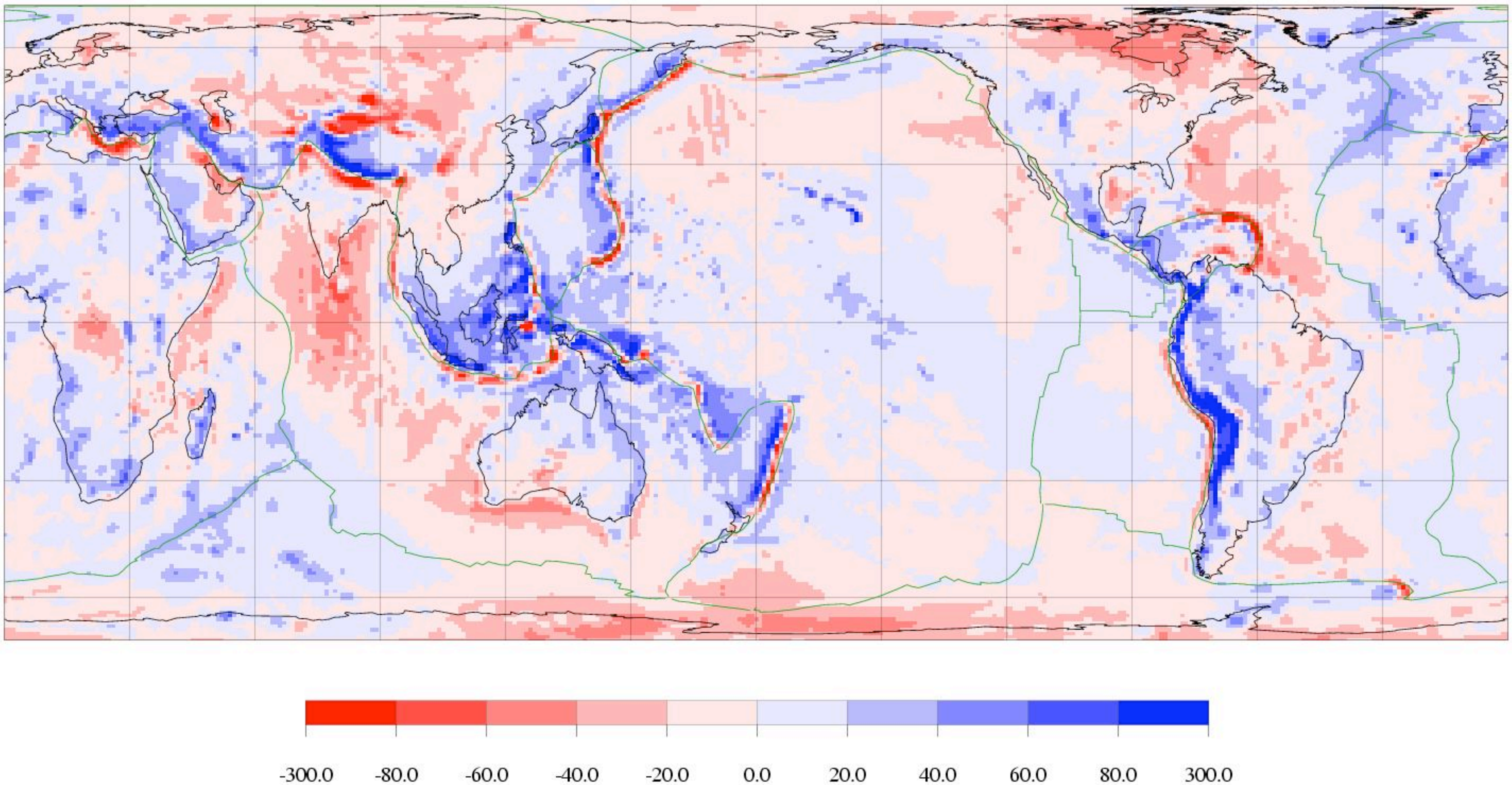
Gravity anomaly (mGal) corresponding to C



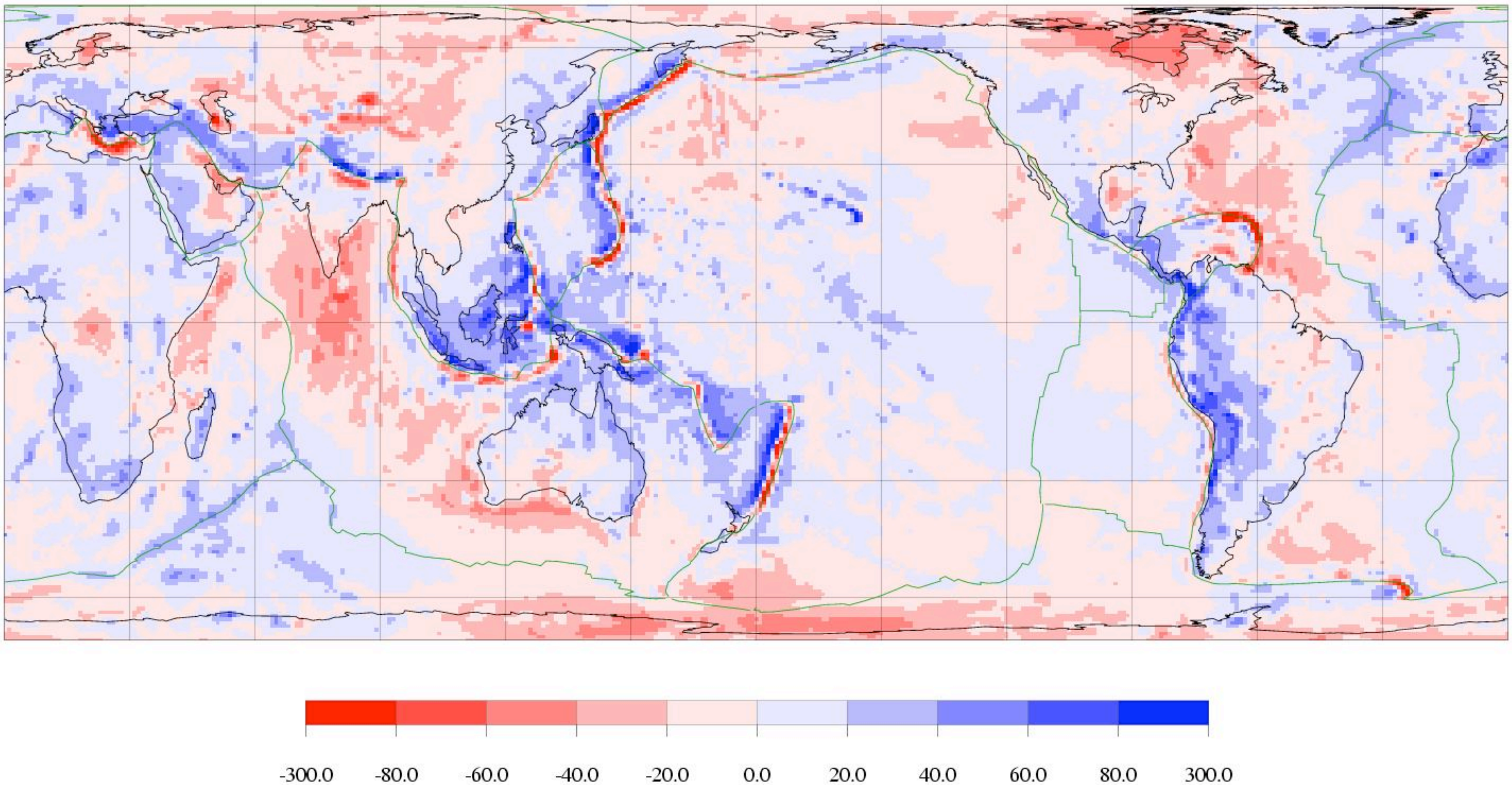
Gravity anomaly (mGal) corresponding to D



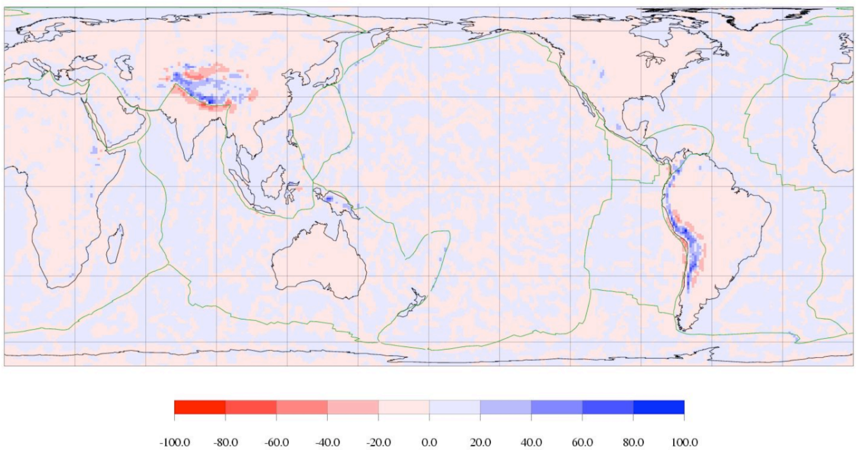
Gravity anomaly (mGal) corresponding to N



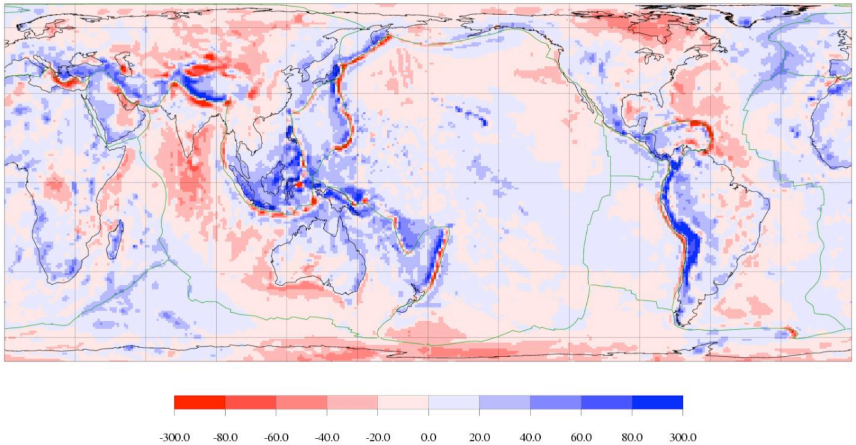
Gravity anomaly (mGal) corresponding to EGM96



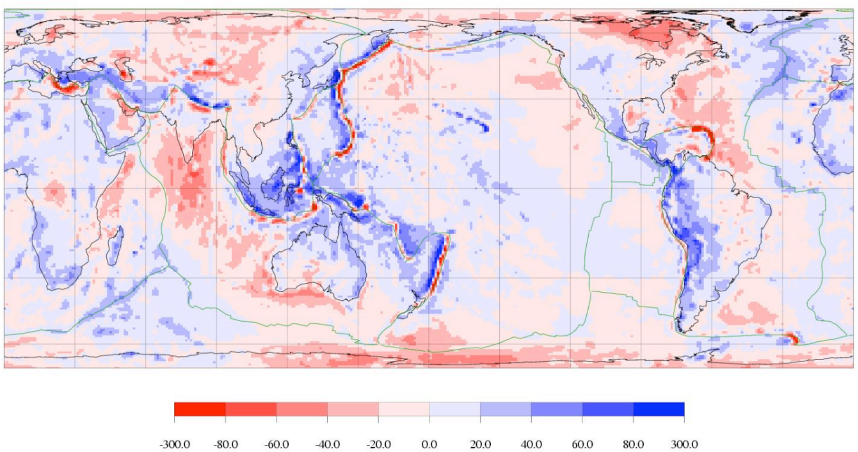
Gravity anomaly (mGal) corresponding to EGM96-N



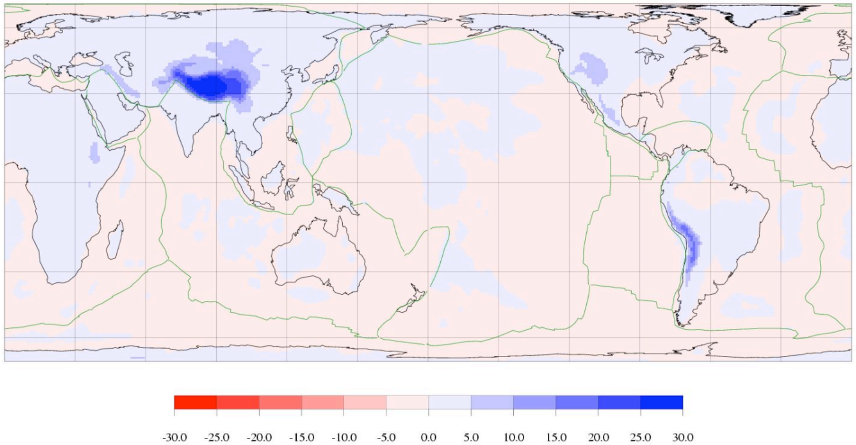
Gravity anomaly (mGal) corresponding to N



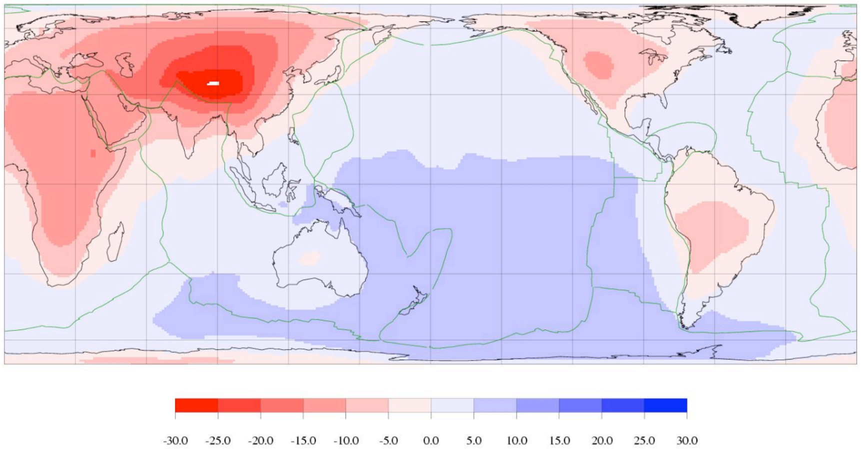
Gravity anomaly (mGal) corresponding to EGM96



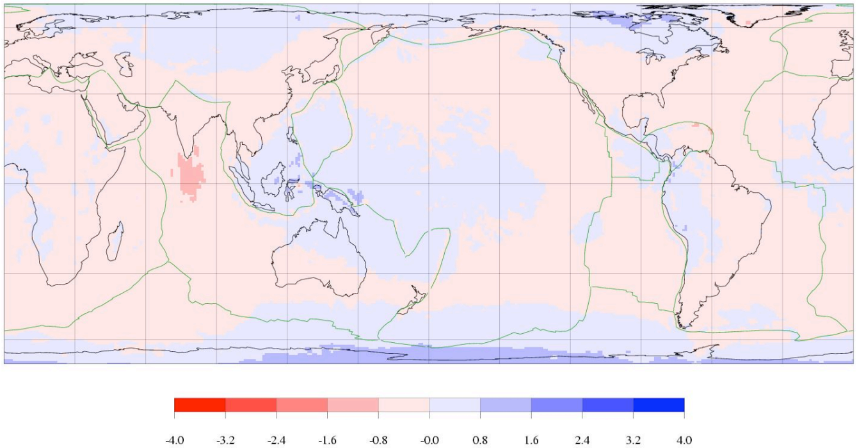
Gravity anomaly (mGal) corresponding to EGM96-N



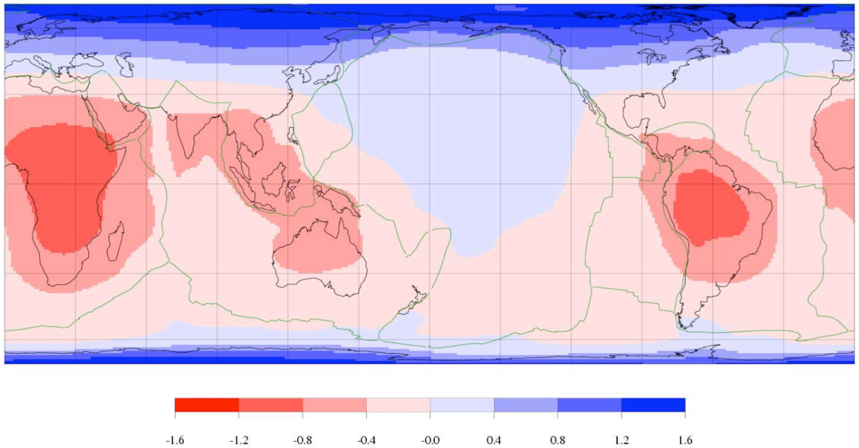
Height anomaly (m) corresponding to A



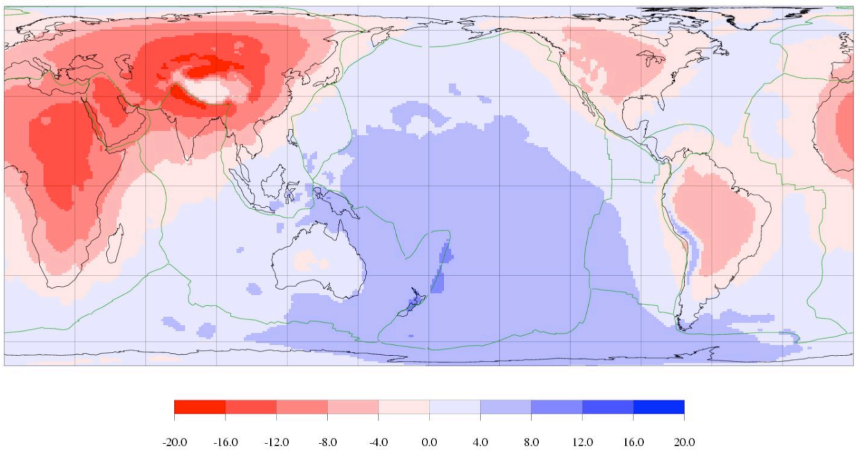
Height anomaly (m) corresponding to B



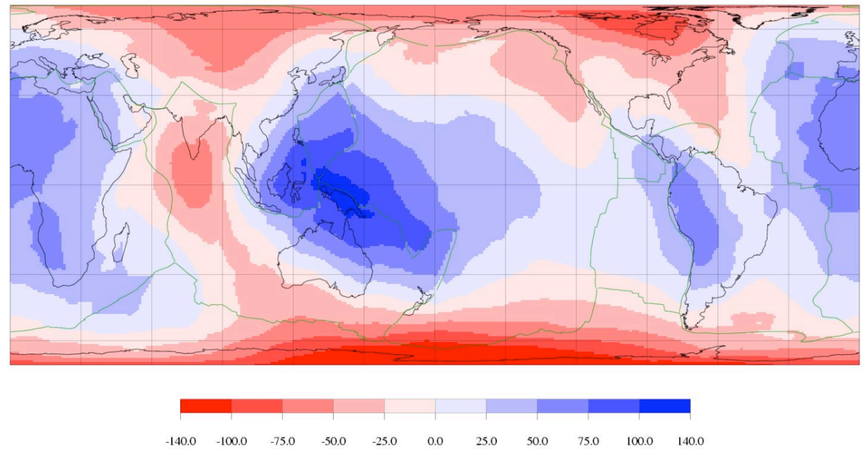
Height anomaly (m) corresponding to C



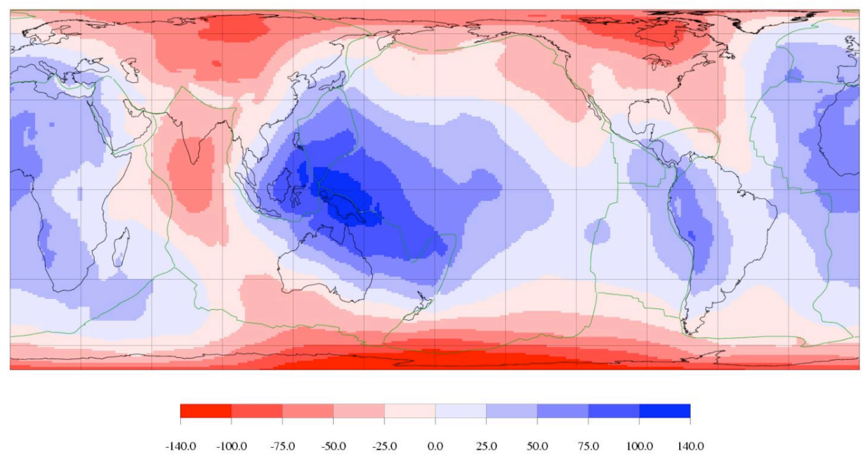
Height anomaly (m) corresponding to D



Height anomaly (m) corresponding to N



Height anomaly (m) corresponding to EGM96-N



Height anomaly (m) corresponding to EGM96

Produit scalaire

$$< \zeta_{\rm obs}, \zeta_A >_\ell = \sum_{m=-\ell}^{\ell} \zeta_{\rm obs}{}_\ell^m \zeta_A{}_\ell^m.$$

Rapport des spectres

$$\frac{||\zeta_A||_\ell}{||\zeta_{\rm obs}||_\ell} = \frac{\sqrt{< \zeta_A, \zeta_A >_\ell}}{\sqrt{< \zeta_{\rm obs}, \zeta_{\rm obs} >_\ell}},$$

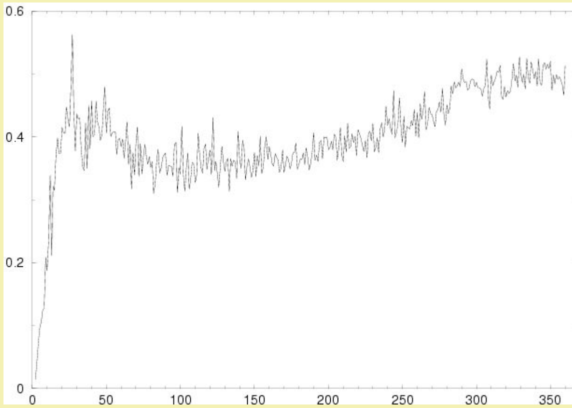
Corrélation

$$Cor_\ell(\zeta_{\rm obs}, \zeta_A) = \frac{< \zeta_{\rm obs}, \zeta_A >_\ell}{||\zeta_{\rm obs}||_\ell ||\zeta_A||_\ell},$$

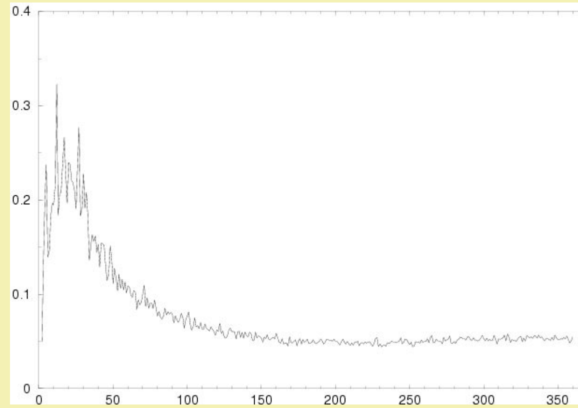
Réduction de variance

$$V_\ell(\zeta_{\rm obs}, \zeta_A) = \frac{||\zeta_{\rm obs}||_\ell^2 - ||\zeta_{\rm obs} - \zeta_A||_\ell^2}{||\zeta_{\rm obs}||_\ell^2}.$$

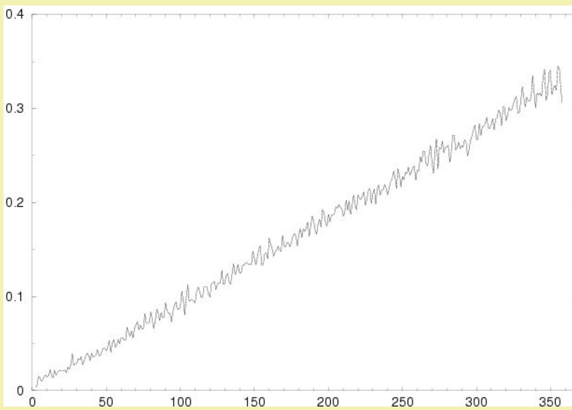
A



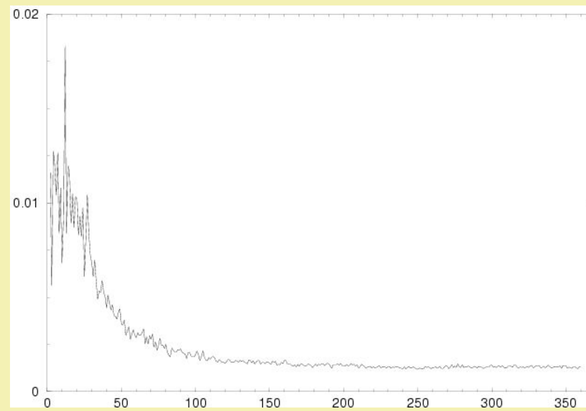
B



C

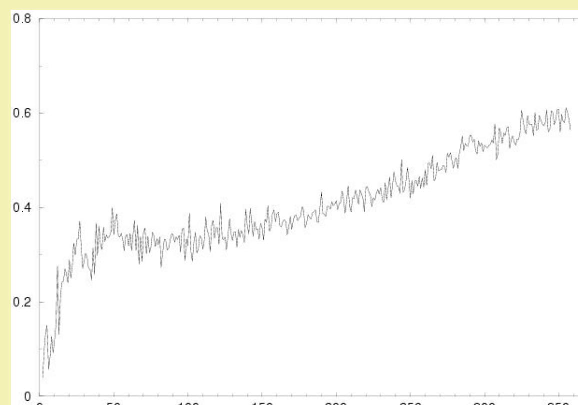


D

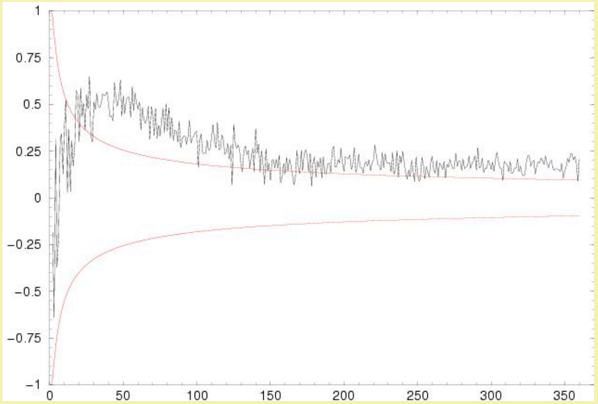


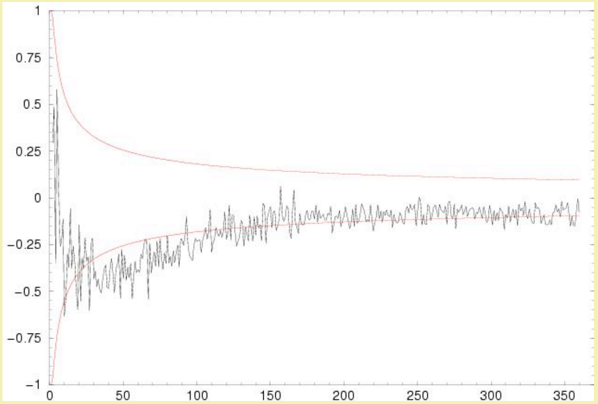
Rapport spectraux :
2ème ordre / EGM 96

N



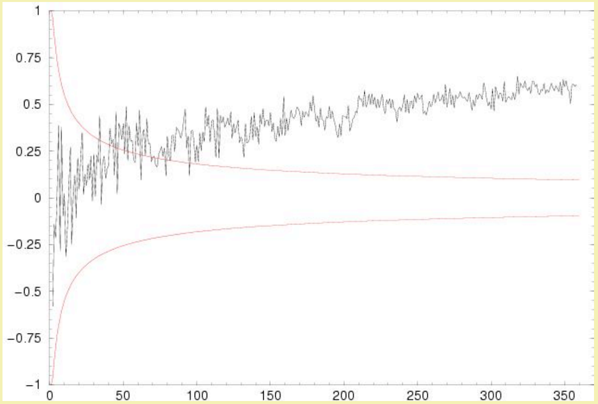
A

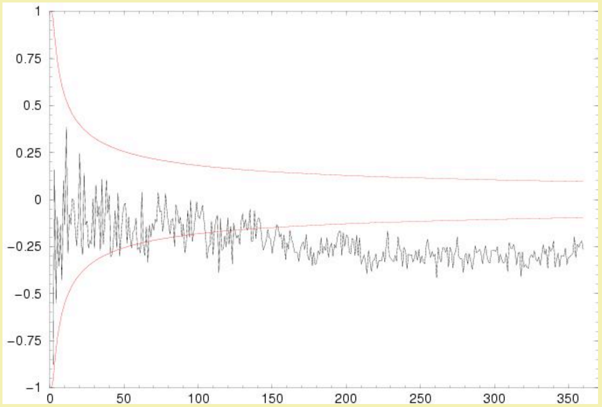




B

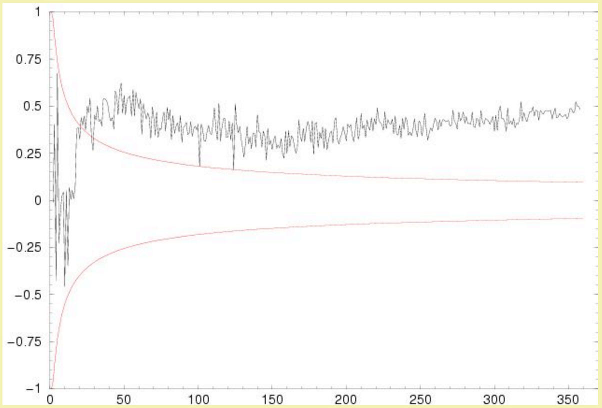
C





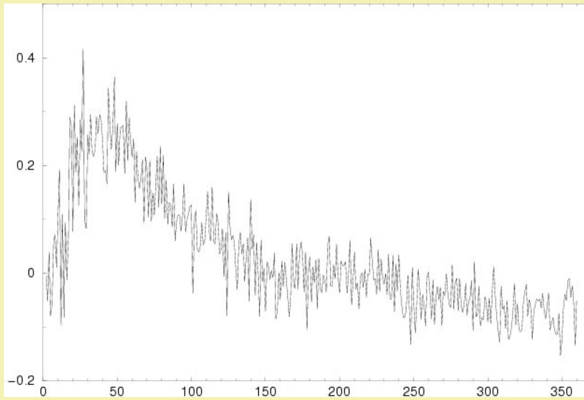
D

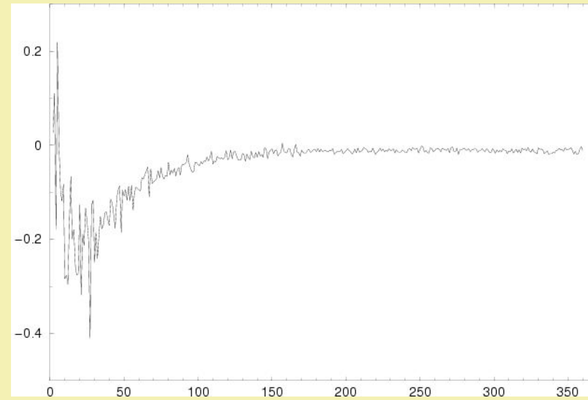
Corrélations :
2ème ordre * EGM 96



N

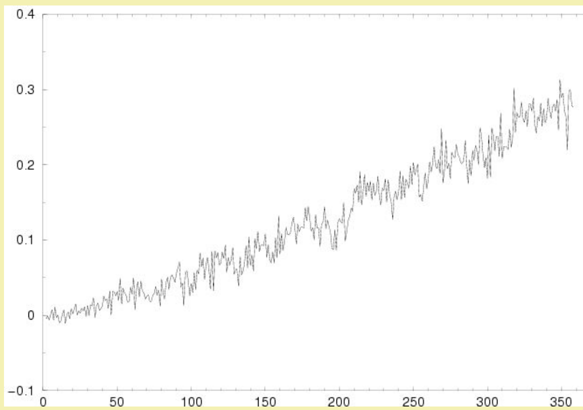
A

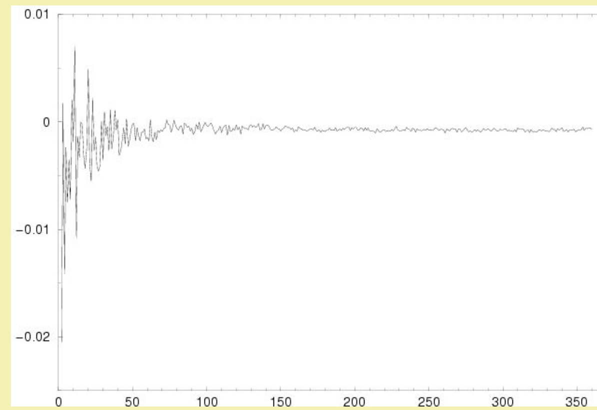




B

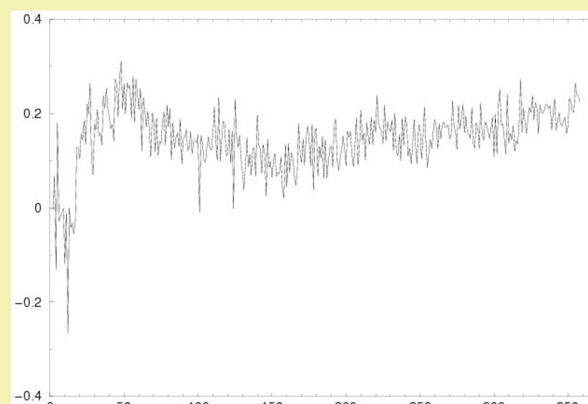
C





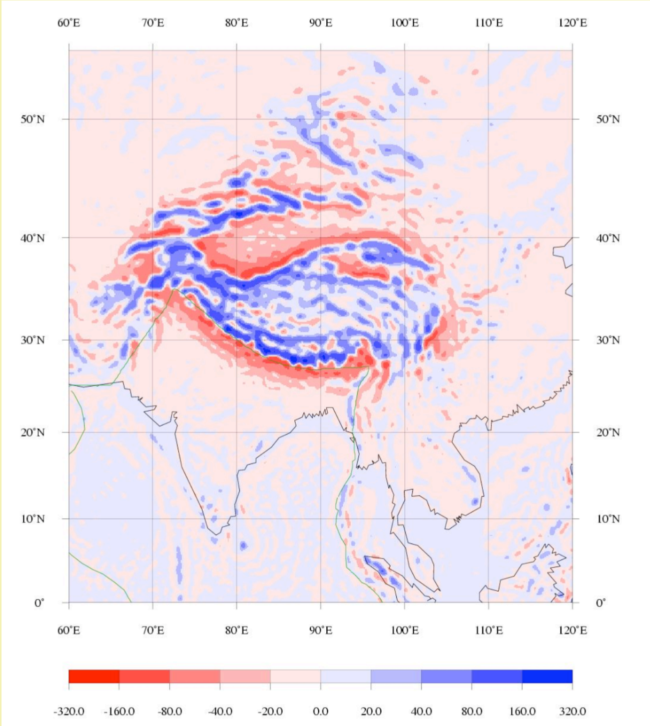
D

Réductions de variance :
2ème ordre - EGM 96



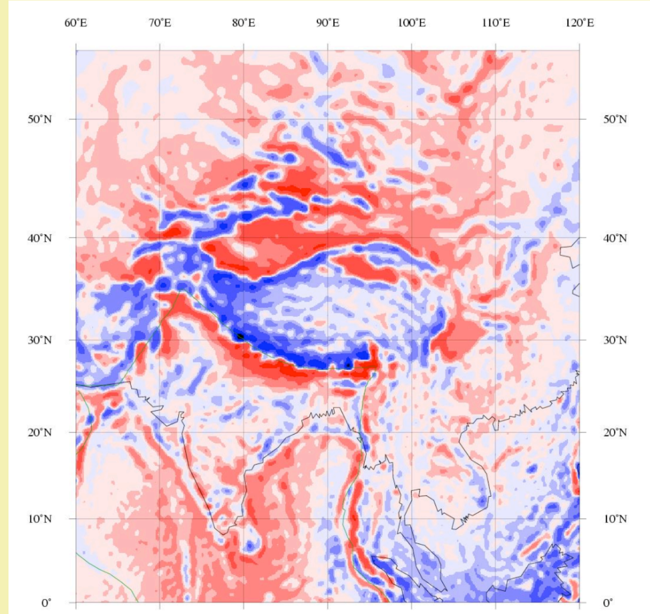
N

N



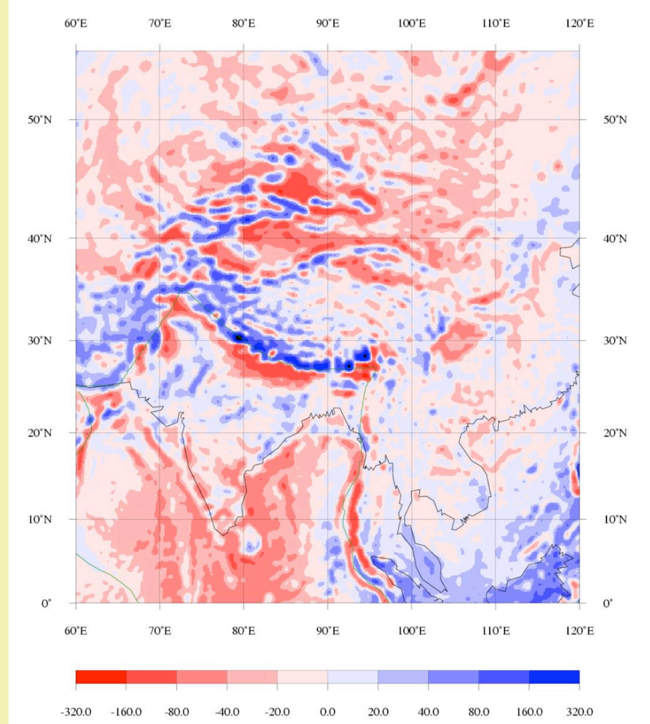
60°E 70°E 80°E 90°E 100°E 110°E 120°E
50°N 40°N 30°N 20°N 10°N 0°
-320.0 -160.0 -80.0 -40.0 -20.0 0.0 20.0 40.0 80.0 160.0 320.0

Obs



60°E 70°E 80°E 90°E 100°E 110°E 120°E
50°N 40°N 30°N 20°N 10°N 0°

Obs-N



60°E 70°E 80°E 90°E 100°E 110°E 120°E
50°N 40°N 30°N 20°N 10°N 0°
-320.0 -160.0 -80.0 -40.0 -20.0 0.0 20.0 40.0 80.0 160.0 320.0

Avez-vous reconnu
cet illustre géodésien
sans sa perruque ?


REPUBLIQUE FRANÇAISE
18 F
POSTES
NEWTON
1642-1727
H LALAU
P. MUNIER


ONE POUND

Annexe A : harmoniques sphériques

$$Y_\ell^m(\theta,\lambda) = \begin{cases} p_\ell^m(\cos\theta)\cos(m\lambda) & \text{if } m \geq 0, \\ p_\ell^{|m|}(\cos\theta)\sin(|m|\lambda) & \text{if } m < 0 \end{cases}$$

$$\frac{1}{4\pi}\int_\Omega Y_\ell^m(\theta,\lambda)\,Y_{\ell'}^{m'}(\theta,\lambda)\,\mathrm{d}\Omega = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} Y_\ell^m(\theta,\lambda)\,Y_{\ell'}^{m'}(\theta,\lambda)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\lambda = \delta_\ell^{\ell'}\,\delta_m^{m'},$$

$$Y_0^0(\theta,\lambda) = 1, \quad Y_1^0(\theta,\lambda) = \sqrt{3}\cos\theta, \quad Y_1^1(\theta,\lambda) = \sqrt{3}\sin\theta\cos\lambda, \quad Y_1^{-1}(\theta,\lambda) = \sqrt{3}\sin\theta\sin\lambda,$$

$$Y_2^0(\theta,\lambda) = \frac{\sqrt{5}}{2}(3\cos^2\theta - 1).$$

$$h_\ell^m = \frac{1}{4\pi} \int_\Omega h(\theta, \lambda) Y_\ell^m(\theta, \lambda) \, \mathrm{d}\Omega,$$

$$h(\theta, \lambda) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} h_\ell^m \, Y_\ell^m(\theta, \lambda).$$

$$\int_{V_0} h Y_\ell^m \, \mathrm{d}V = \int_0^b \int_\Omega h Y_\ell^m \, \mathrm{d}\Omega \, \mathrm{d}r = 4\pi \int_0^b h_\ell^m(r) r^2 \, \mathrm{d}r.$$

Annexe B - Perturbations of $\int_V f(x) r^k Y_\ell^m \mathrm{d}V$

$$\mathcal{F} = \int_V f(x) r^k Y_\ell^m \, \mathrm{d}V$$

$$\begin{aligned} \delta_1 \mathcal{F} &= \int_{V_0} r^k \delta_{1e} f Y_\ell^m \, \mathrm{d}V - \int_{\Sigma_0} r^k [f \boldsymbol{\xi}_1 \cdot \boldsymbol{n}] Y_\ell^m \, \mathrm{d}\Sigma, \\ &= \int_{V_0} \left(r^k \delta_{1l} f + f \mathrm{div}(r^k \boldsymbol{\xi}_1) \right) Y_\ell^m \, \mathrm{d}V, \end{aligned}$$

Annexe C - Expression of C_ℓ^m

$$C_\ell^m = -\frac{2}{3\sqrt{5}}\epsilon(b)(\ell+3)\int_\Omega X_\ell Y_2^0 Y_\ell^m \,\mathrm{d}\Omega$$

$$L_\ell^m = \int_\Omega X_\ell Y_\ell^m \,\mathrm{d}\Omega.$$

$$X_\ell = \int_0^b \delta_d \rho\, r^{\ell+2}\,\mathrm{d}r - \sum_{r_\Sigma}[\rho]\xi_d r_\Sigma^{\ell+2}.$$

$$Y_2^0 Y_\ell^m = \sum_{\ell' m'} \gamma_{\ell\ \ell'\ 2}^{m m' 0} Y_{\ell'}^{m'}, \qquad \gamma_{\ell\ \ell'\ s}^{m m' t} = \frac{1}{4\pi}\int_\Omega Y_{\ell'}^{m'} Y_\ell^m Y_s^t \,\mathrm{d}\Omega.$$

$$\int_\Omega h Y_2^0 Y_\ell^m \,\mathrm{d}\Omega = 4\pi \sum_{\ell' m'} \gamma_{\ell\ \ell'\ 2}^{m m' 0}\ h_{\ell'}^{m'},$$

$$Y_2^0 Y_\ell^m = \gamma_{\ell\ \ell-2\,2}^{m\,m\ \ 0} Y_{\ell-2}^m + \gamma_{\ell\ \ell\ 2}^{m\,m\,0} Y_\ell^m + \gamma_{\ell\ \ell+2\,2}^{m\,m\ \ 0} Y_{\ell+2}^m,$$

$$\gamma_{\ell\ \ell'\ 2}^{m\,m'\,0} = (-1)^m \sqrt{5(2\ell+1)(2\ell'+1)} \begin{pmatrix} \ell & \ell' & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell' & 2 \\ -m & m' & 0 \end{pmatrix}.$$

$$\gamma_{\ell\ \ell\ 2}^{m\,m\,0} = \sqrt{5}\,\frac{\ell(\ell+1)-3m^2}{(2\ell-1)(2\ell+3)},$$

$$\gamma_{\ell\ \ell-2\,2}^{m\,m\quad 0} = \frac{3}{2}\sqrt{5}\left(\frac{(\ell-m)(\ell-m-1)(\ell+m)(\ell+m-1)}{(2\ell-3)(2\ell-1)^2(2\ell+1)}\right)^{\frac{1}{2}},$$

$$\gamma_{\ell\ \ell+2\,2}^{m\,m\quad 0} = \frac{3}{2}\sqrt{5}\left(\frac{(\ell-m+2)(\ell-m+1)(\ell+m+2)(\ell+m+1)}{(2\ell+1)(2\ell+3)^2(2\ell+5)}\right)^{\frac{1}{2}}.$$

$$C_\ell^m = -\frac{2}{3\sqrt{5}}\epsilon(b)(\ell+3)\left\{\gamma_{\ell\ \ell-2\ 2}^{m\ m\ \ 0}\int_\Omega X_\ell Y_{\ell-2}^m \,\mathrm{d}\Omega + \gamma_{\ell\ \ell\ 2}^{m\,m\,0}\int_\Omega X_\ell Y_\ell^m \,\mathrm{d}\Omega + \gamma_{\ell\ \ell+2\ 2}^{m\ m\ \ 0}\int_\Omega X_\ell Y_{\ell+2}^m \,\mathrm{d}\Omega\right\}.$$

$$C_\ell^m = -\frac{2}{3\sqrt{5}}\epsilon(b)(\ell+3)\left\{\gamma_{\ell\ \ell-2\ 2}^{m\ m\ \ 0}\, b^2 L_{\ell-2}^m + \gamma_{\ell\ \ell\ 2}^{m\,m\,0}\, L_\ell^m + \gamma_{\ell\ \ell+2\ 2}^{m\ m\ \ 0}\, b^{-2} L_{\ell+2}^m\right\}.$$

$$L_\ell^m = \frac{4\pi\rho_2}{3} b^{\ell+3}\left(2\frac{\delta_d g_\ell^m}{g} + 3\frac{\zeta_{d\,\ell}^{\,m}}{b}\right) = \frac{4\pi\rho_2}{3} b^{\ell+3} Z_\ell^m,$$

$$C_\ell^m = -\frac{8\pi\rho_2}{9\sqrt{5}}\epsilon(b)(\ell+3)b^{\ell+3}\left(\gamma_{\ell\ \ell-2\ 2}^{m\ m\ \ 0}\, Z_{\ell-2}^m + \gamma_{\ell\ \ell\ 2}^{m\,m\,0}\, Z_\ell^m + \gamma_{\ell\ \ell+2\ 2}^{m\ m\ \ 0}\, Z_{\ell+2}^m\right).$$

$$\zeta_{C\,\ell}^{\,m} = -\frac{2\epsilon(b)}{3\sqrt{5}}\frac{\ell+3}{2\ell+1} b\left(\gamma_{\ell\ \ell-2\ 2}^{m\ m\ \ 0}\, Z_{\ell-2}^m + \gamma_{\ell\ \ell\ 2}^{m\,m\,0}\, Z_\ell^m + \gamma_{\ell\ \ell+2\ 2}^{m\ m\ \ 0}\, Z_{\ell+2}^m\right) r^{\ell+2} \approx r^\ell \cdot b^2$$

$$\zeta_{C\,\ell}^{\,m} = -\frac{\epsilon(b)}{6\pi\sqrt{5}}\frac{\ell+3}{2\ell+1} b\int_\Omega Z Y_2^0 Y_\ell^m \,\mathrm{d}\Omega,$$

Autres formules

$$\delta_{1l} f = \delta_{1e} f + \textbf{grad}\ f \cdot \boldsymbol{\xi}_1,$$

$$\int_V \boldsymbol{u} \cdot \textbf{grad}\ f \,\mathrm{d}V = -\int_V f\ \mathrm{div}\ \boldsymbol{u} \,\mathrm{d}V - \int_\Sigma [f \boldsymbol{u} \cdot \boldsymbol{n}] \,\mathrm{d}\Sigma,$$