Le signal du marégraphe de Brest vu par les ondelettes.

A. Rivoldini et F. Renaud. Observatoire Royal de Belgique.

November 13, 2003

Why use wavelets?

- time-scale ($scale \sim 1/frequency$) representation
  - characterization of the components of finite length
  - characterization of transients
  - instantaneous scale and amplitude determination
- local transform $\rightarrow$ robust representation
- ...

Where do wavelets come from?

- Not based on a ”bright new idea” but on concepts that already existed under various forms in different fields
- Alex Grossmann (quantum mechanics) and Jean Morlet (seismology) in 1981 formalized the continuous wavelet transform
- Wavelet frames I. Daubechies in 1985
- Multiresolution signal approximation S. Mallat 1989

Some wavelet-related applications

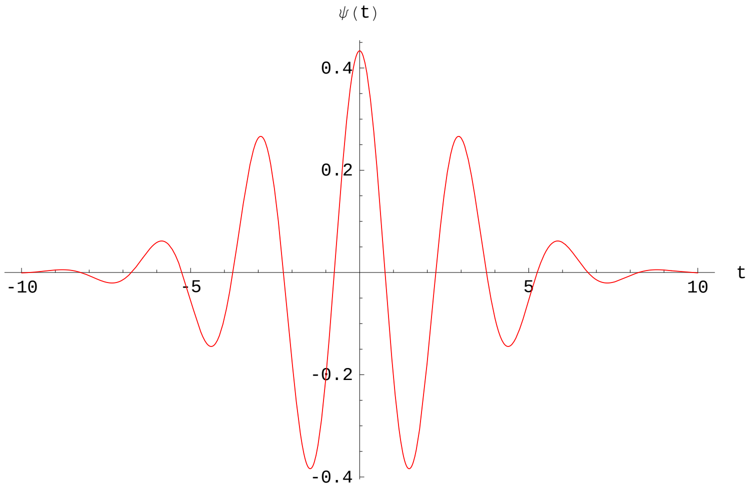
- signal processing (1d, 2d,3d,2d+1, sphere,..)
- signal denoising by non-linear smoothing
- image processing (directional filtering, feature extraction,..)
- data compression (JPEG 2000,..)
- numerical resolution of partial differential equations
- pure math (function space and operator characterization)
- ...

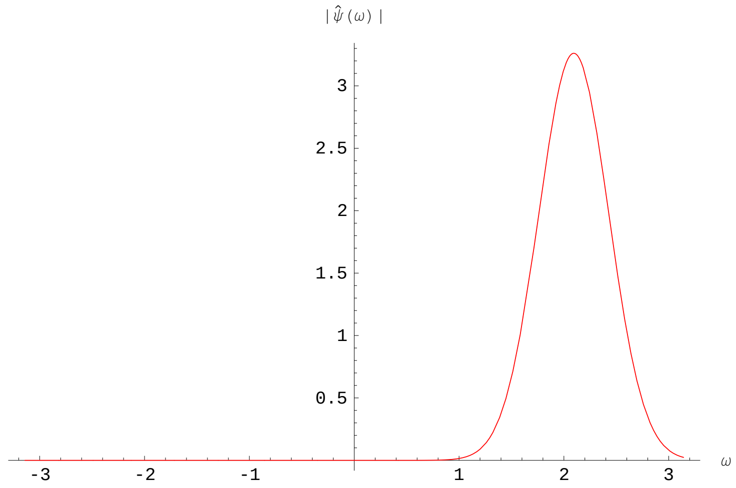
What is a wavelet?

- a function $\psi \in \mathbf{L^2}(\mathbb{R})$
- of zero mean $\int_{-\infty}^{+\infty} dt\, \psi(t) = 0$ or $\hat{\psi}(0) = 0$
- normalized $||\psi||_{\mathbf{L^2}} = 1$
- centered in the neighborhood of $t = 0$
- analytic if $\hat{\psi}(\omega) = 0$ for $\omega < 0$
- with n zero moments if $\int_{-\infty}^{+\infty} dt\, t^n(t)\, \psi(t) = 0$

Example: Morlet wavelet

$$\psi(t) = \frac{1}{(\sigma^2\pi)^{1/4}}\, e^{i\omega_0 t} e^{-\frac{t^2}{2\sigma^2}} \qquad \hat{\psi}(\omega) = (4\pi\sigma^2)^{1/4}\, e^{-\frac{\sigma^2}{2}(\omega-\omega_0)^2}$$

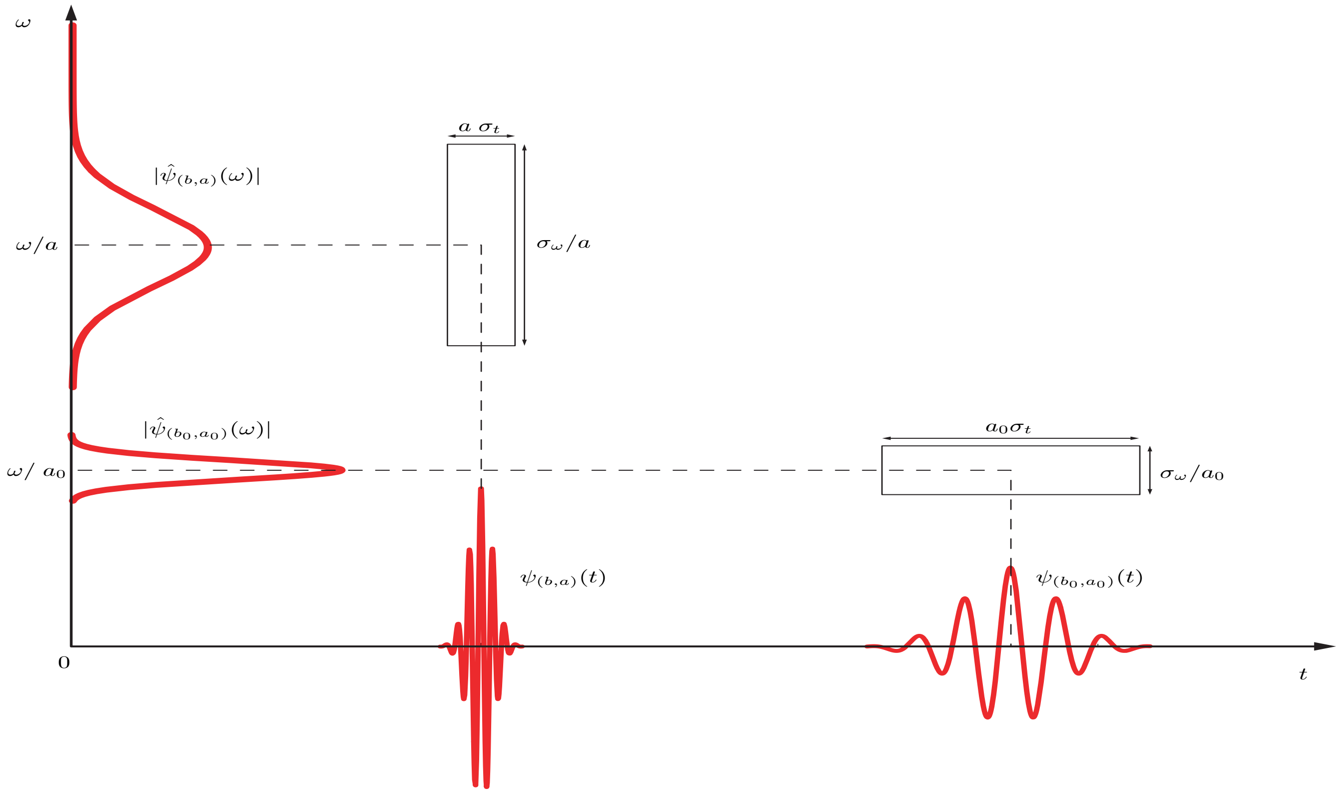




Time-Scale representation.

- create a family of time-scale atoms: scale ψ by $a \in \mathbb{R}^0_+$ and translate it by $b \in \mathbb{R}$:

$$\psi_{(b,a)}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right) \qquad ||\psi_{(b,a)}|| = 1 \qquad \hat{\psi}_{(b,a)}(\omega) = \sqrt{a}\, e^{-i\omega b}\hat{\psi}(a\omega)$$



The continuous wavelet transform of a function $f \in \mathbf{L^2}(\mathbb{R})$

- project f on the family of time-scale atoms: $T_f(b,a) =< f, \psi_{(b,a)} >$

$$T_f(b,a) = \int_{-\infty}^{+\infty} dt\ f(t)\ \frac{1}{\sqrt{a}}\psi^*(\frac{t-b}{a}) = f * \bar{\psi}_a(b)$$

where

$$\bar{\psi}_a(t) = \frac{1}{\sqrt{a}}\psi^*(\frac{-t}{a}) \text{ and } \hat{\bar{\psi}}_a(\omega) = \sqrt{a}\,\hat{\psi}^*(a\,\omega)$$

or

$$T_f(b,a) = \sqrt{a}\,\mathcal{F}^{-1}[\hat{f}(\omega)\,\hat{\psi}^*(a\,\omega)](b)$$

Implementation

- With Matlab for one scale a:

```
fsig=fft(sig);
```
$\mathcal{F}[f(t)] = \hat{f}(\omega)$

```
omega=pulsvect(length(fsig));
```
$\omega = \{[0,\pi), [-\pi, 0)\}$

```
wave=wavelet(a*omega);
```
$\hat{\psi}(a\,\omega)$

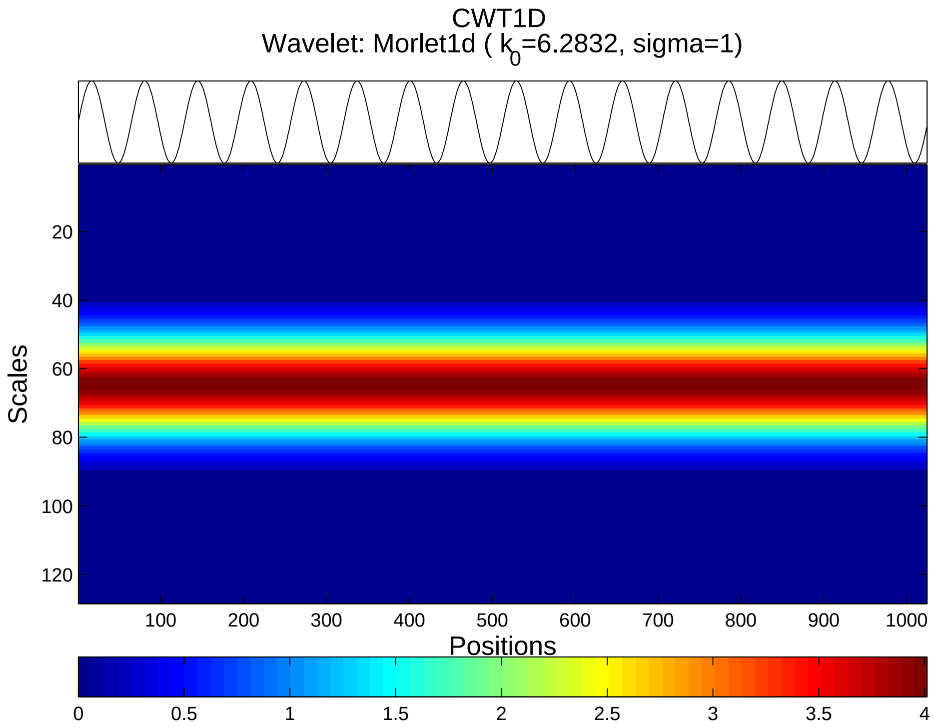
```
cwt=sqrt(a)*ifft(fsig.*conj(wave));
```
$\sqrt{a}\,\mathcal{F}^{-1}[\hat{f}(\omega)\,\hat{\psi}^*(a\,\omega)](b)$

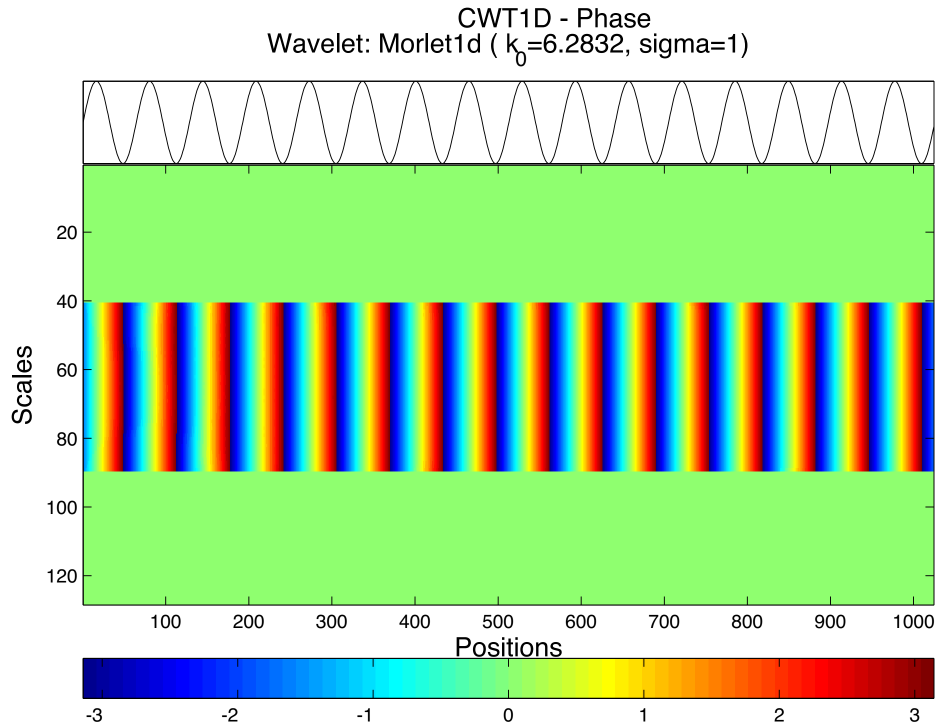
- With YAWTB-Toolbox:

```
sc = vect(1,128 ,128);
```
$a \in [1, 128]$

```
cwt = cwt1d(fft(sig), 'morlet', sc); cwt with Morlet
yashow(wav); shows result on screen
```

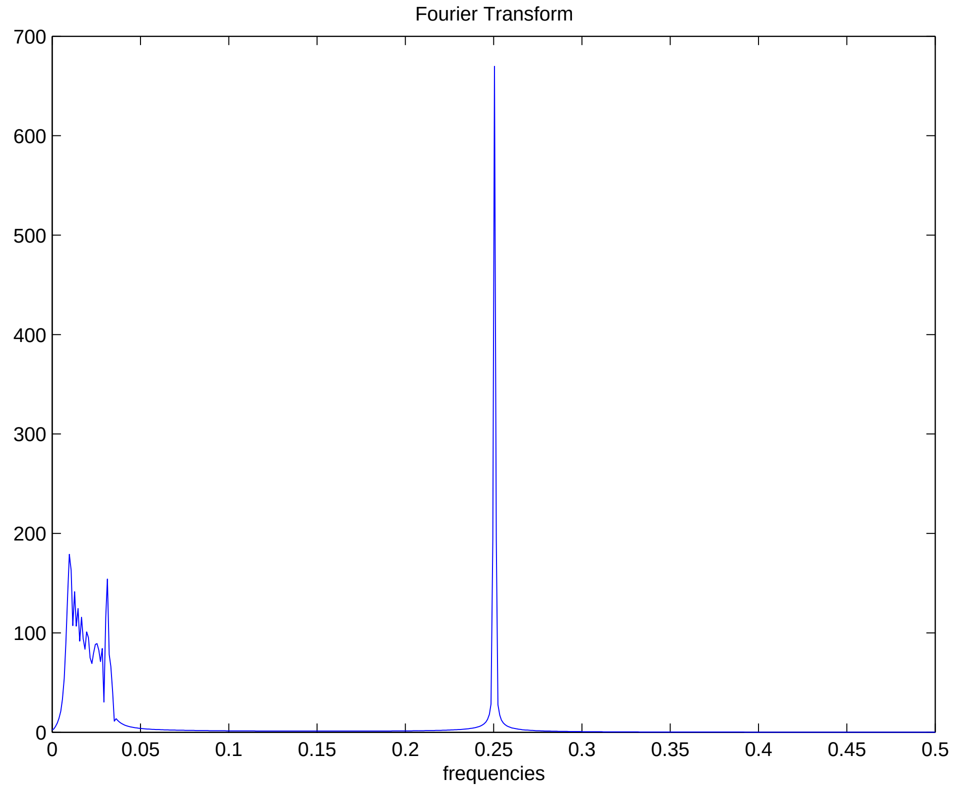
Example: CWT of $\sin(\frac{2\pi}{64}t)$

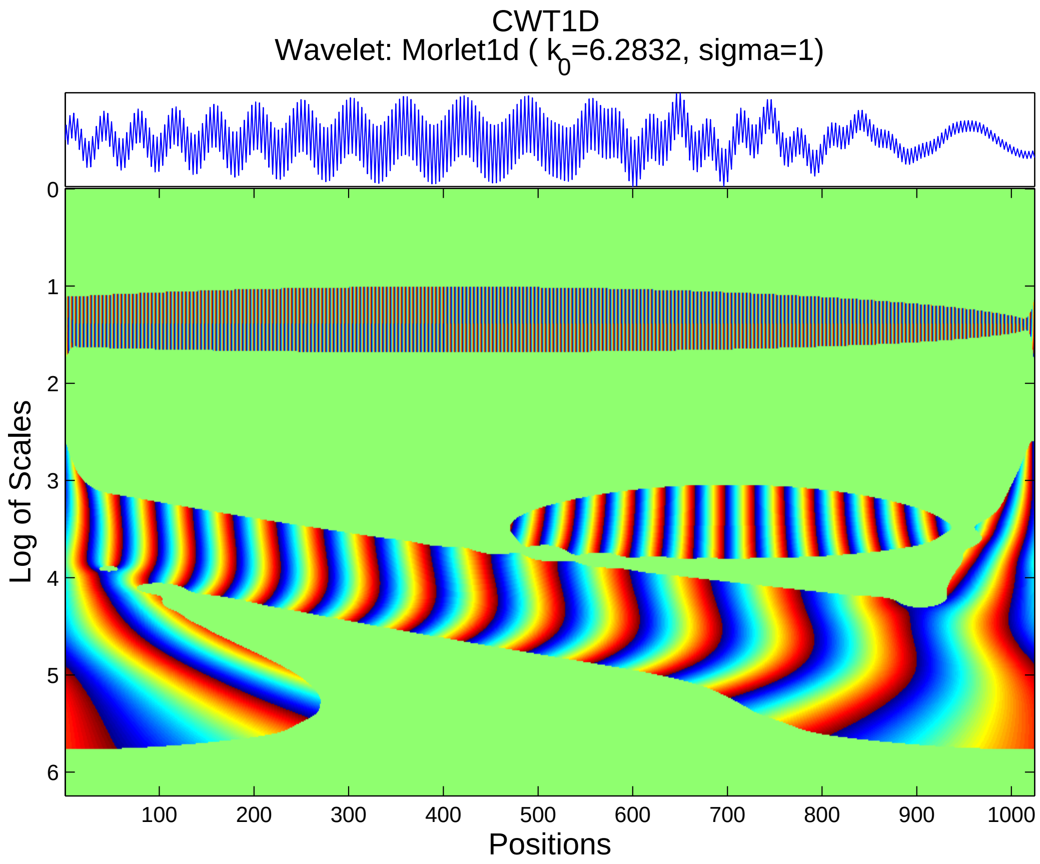




Example: CWT phase

$$f(t) = 2\sin(\frac{2\pi}{4}t)\, e^{-\frac{(t-400)^2}{2\,300^2}} + \sin(\frac{2\pi}{32}t)\, e^{-\frac{(t-700)^2}{2\,100^2}} + \sin(\frac{2\pi}{32}\,\frac{t}{1+t/1000})$$





Wavelet ridges or instantaneous scale

- the instantaneous frequency $\omega(t)$ of a real signal $f(t) = A(t)\cos\phi(t)$ is the positive derivative of its phase: $\omega(t) \equiv \phi'(t) \geq 0$

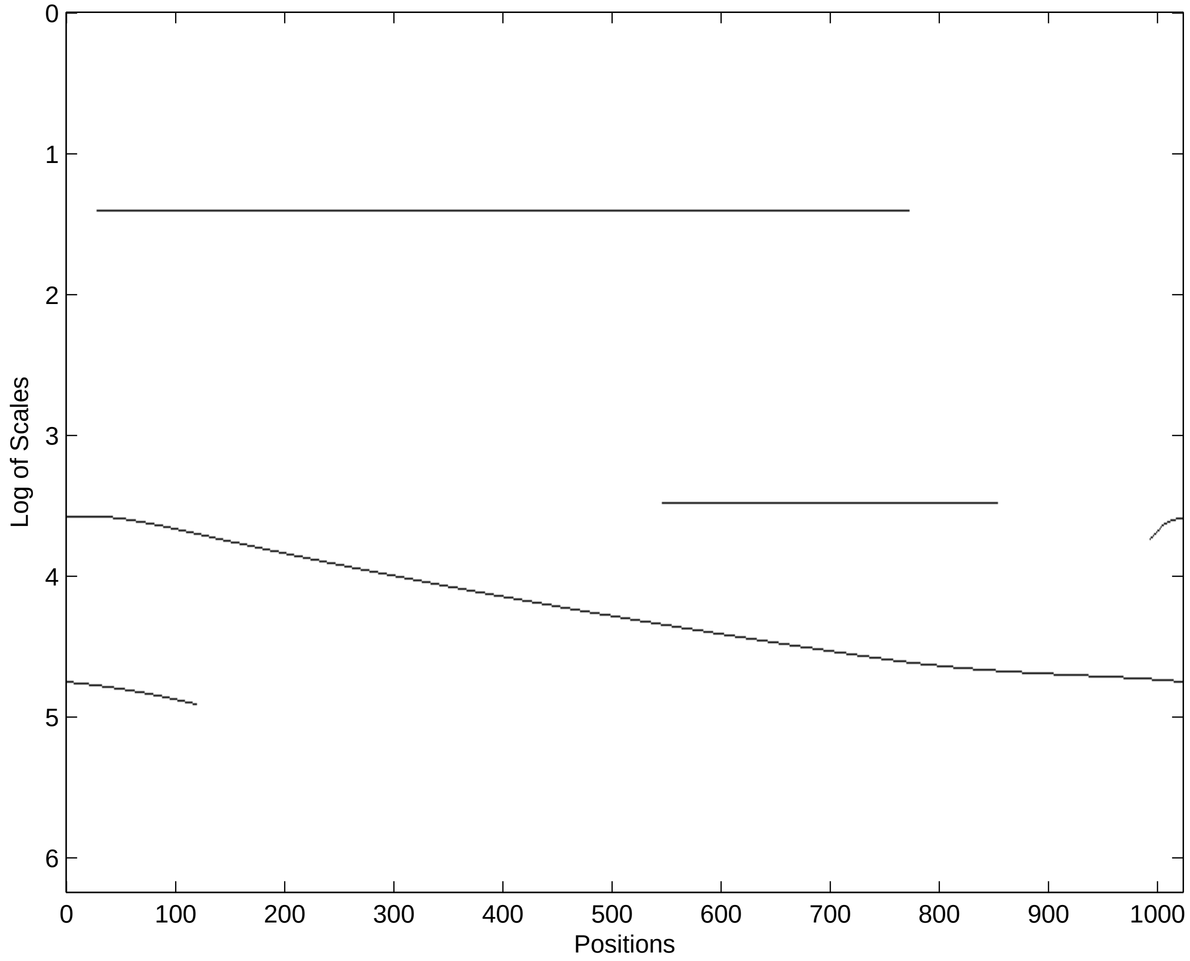
- for an analytic wavelet $\psi_{b,a}(t) = \sqrt{a}\, g(\frac{t-b}{s})e^{i\xi t}e^{-i\xi b}$
$\xi = \frac{\omega_0}{a}$ and g a gaussian with max in ω_0

- the wavelet transform of $f(t)$ is $T_f(b,a) = \frac{\sqrt{a}}{2}A(b)e^{i\phi(b)}(\hat{g}(a[\xi - \phi'(b)]) + ..)$

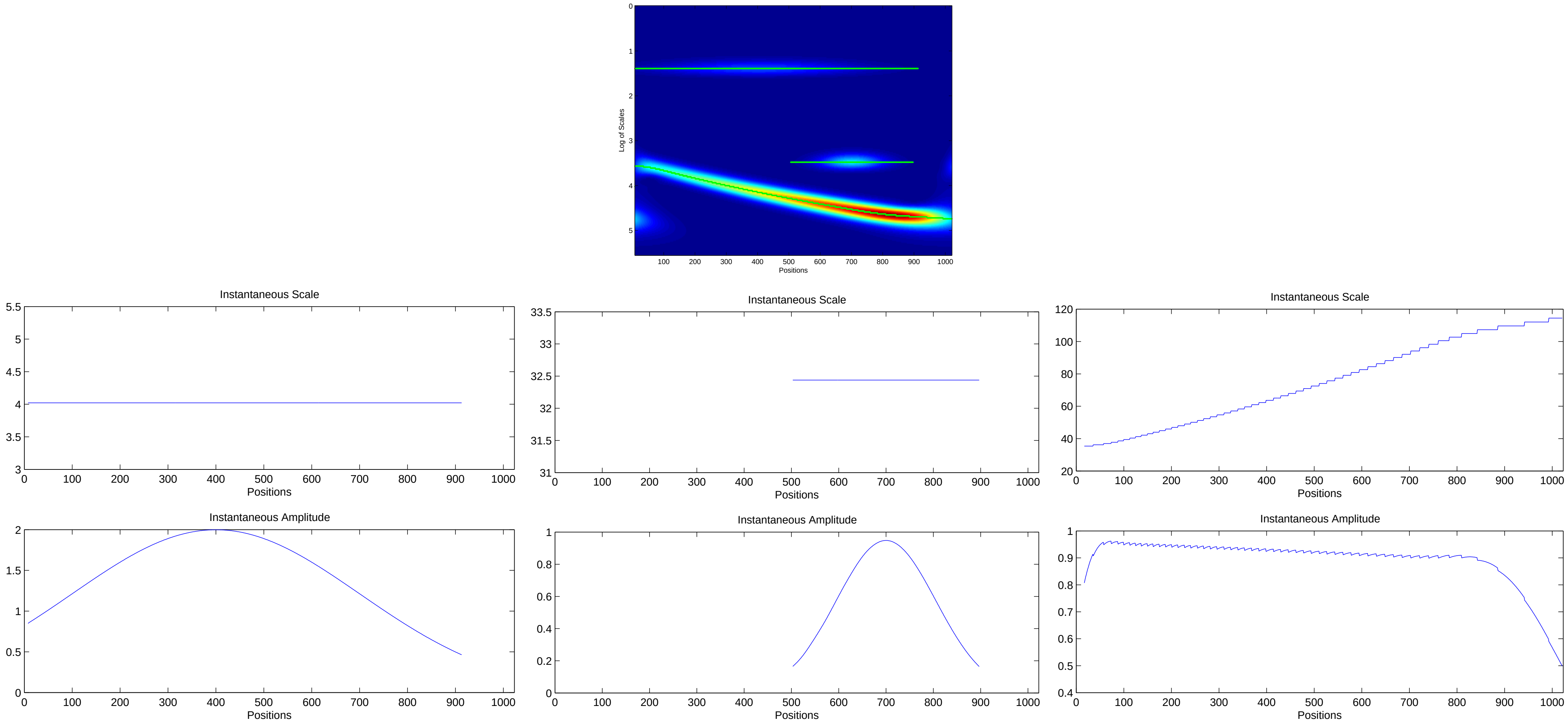
- the wavelet ridge are then the points $(b, \xi(b))$ where the scalogram $|T_f(b,a)|^2$ is maximum. ($\xi(b) = \frac{\omega_0}{a(b)} = \phi'(b)$)

- the analytic amplitude is $A(b) = \frac{2}{\sqrt{a}} \frac{|T_f(b,a(b))|}{|\hat{g}(0)|}$

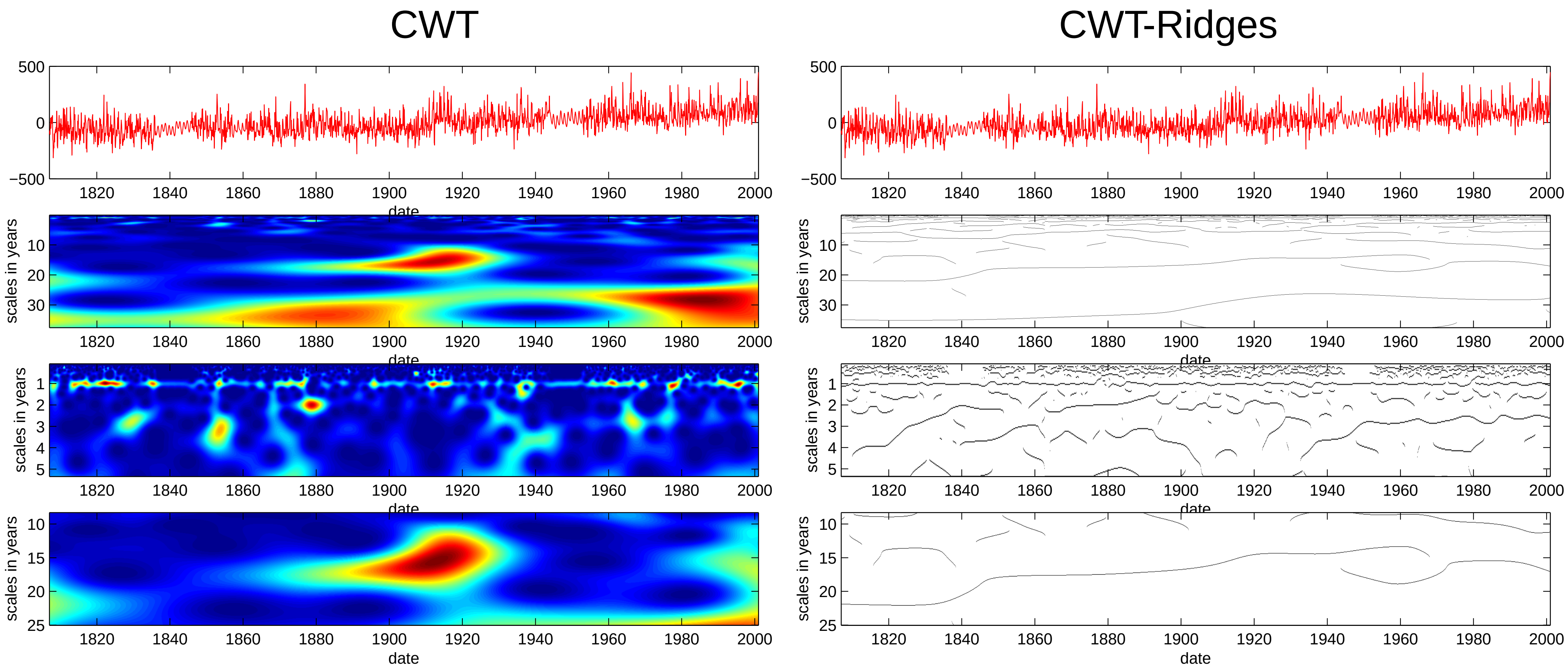
Example: Wavelet - ridges or skeleton

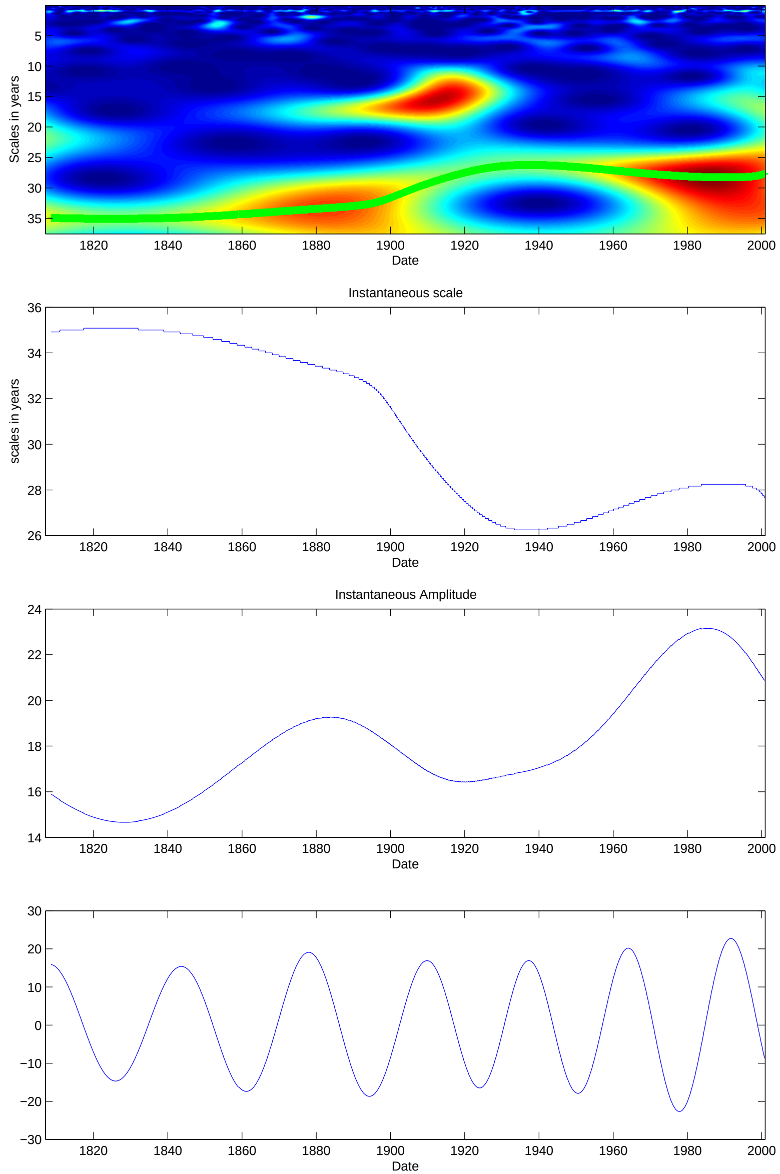


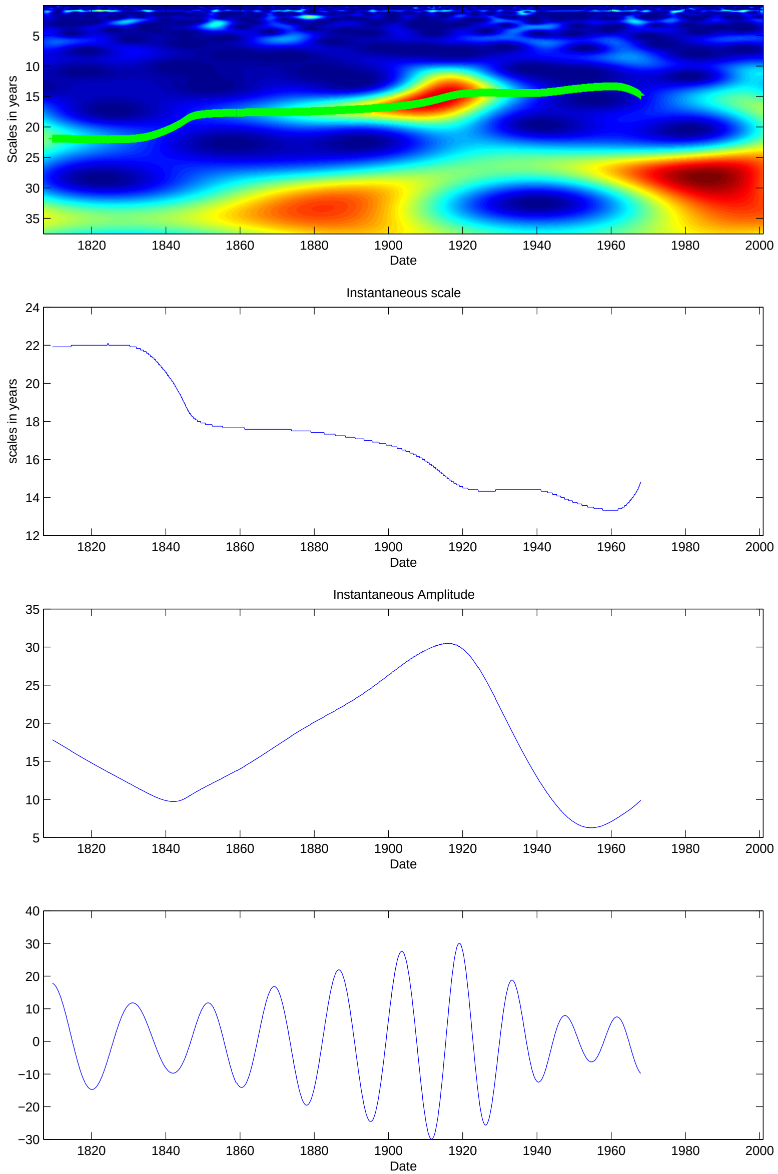
Example: Instantaneous scale and phase

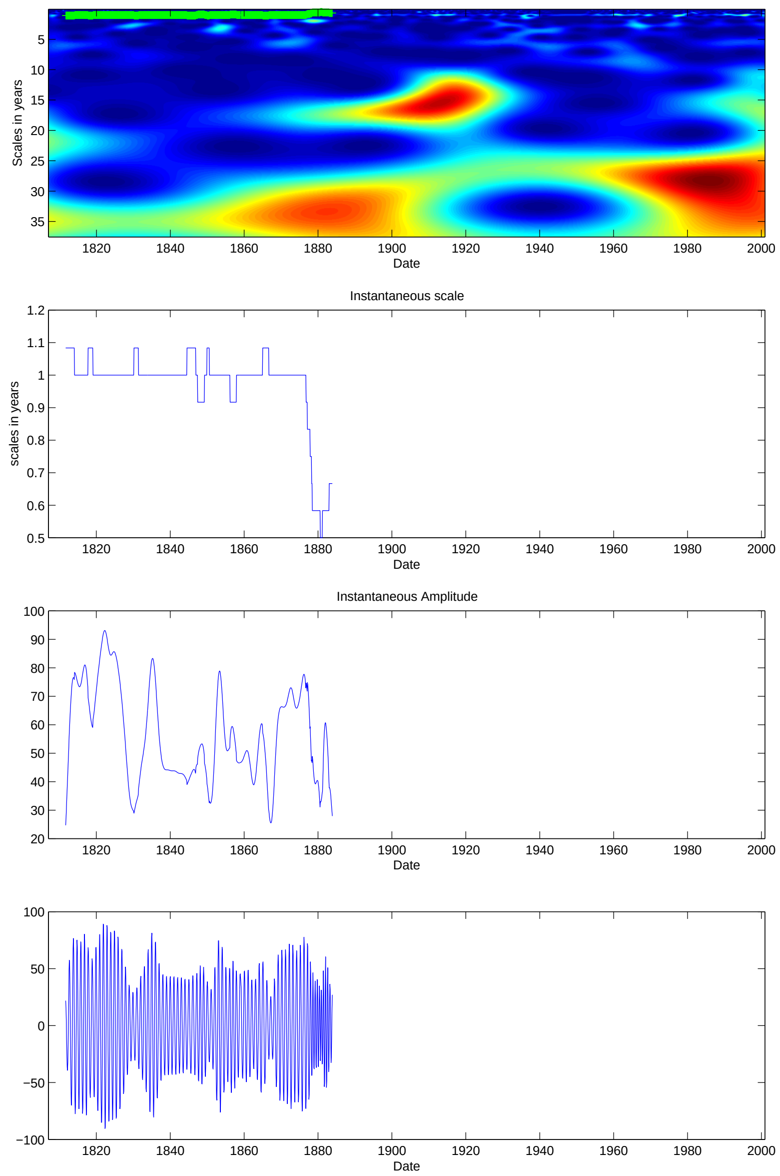


Brest: Whole sampling time - one sample each month - holes filled by .




Scales in years
5
10
15
20
25
30
35
1820
1840
1860
1880
1900
1920
1940
1960
1980
2000
Date
Instantaneous scale
scales in years
36
34
32
30
28
26
Date
Instantaneous Amplitude
24
22
20
18
16
14
Date
30
20
10
0
−10
−20
−30
Date

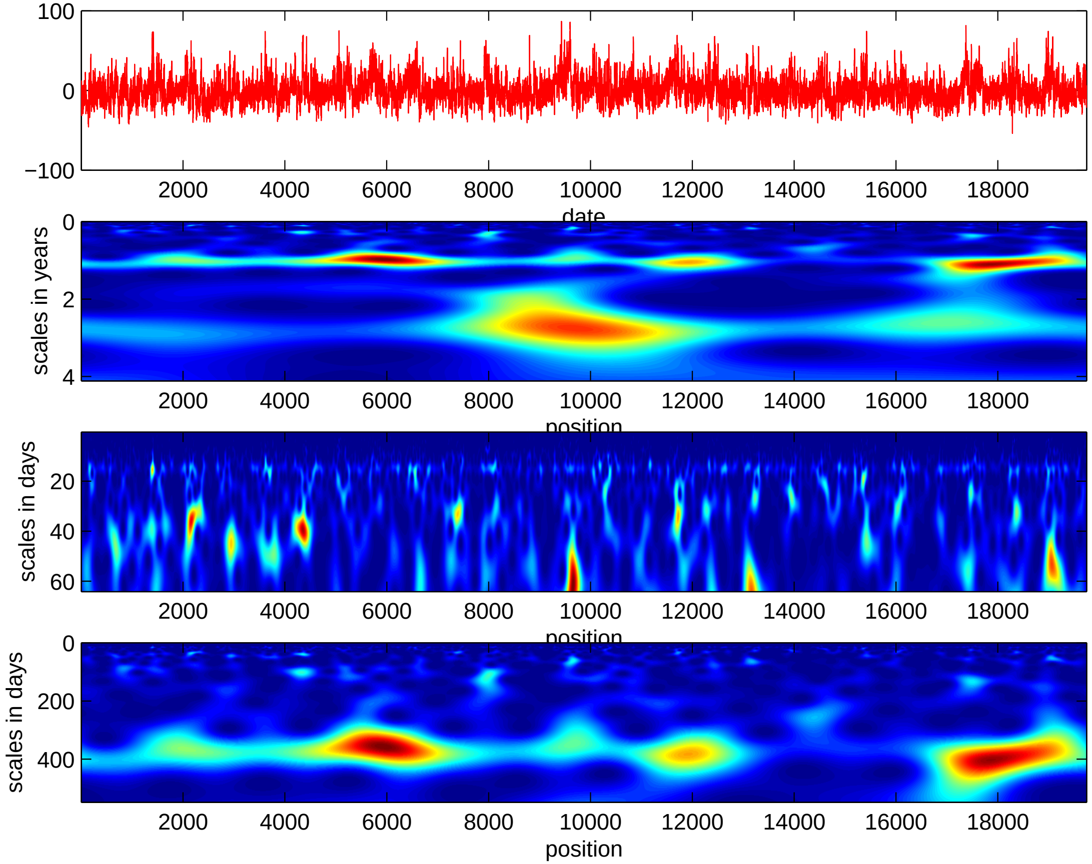

Scales in years
Date
Instantaneous scale
scales in years
Date
Instantaneous Amplitude
Date
Date



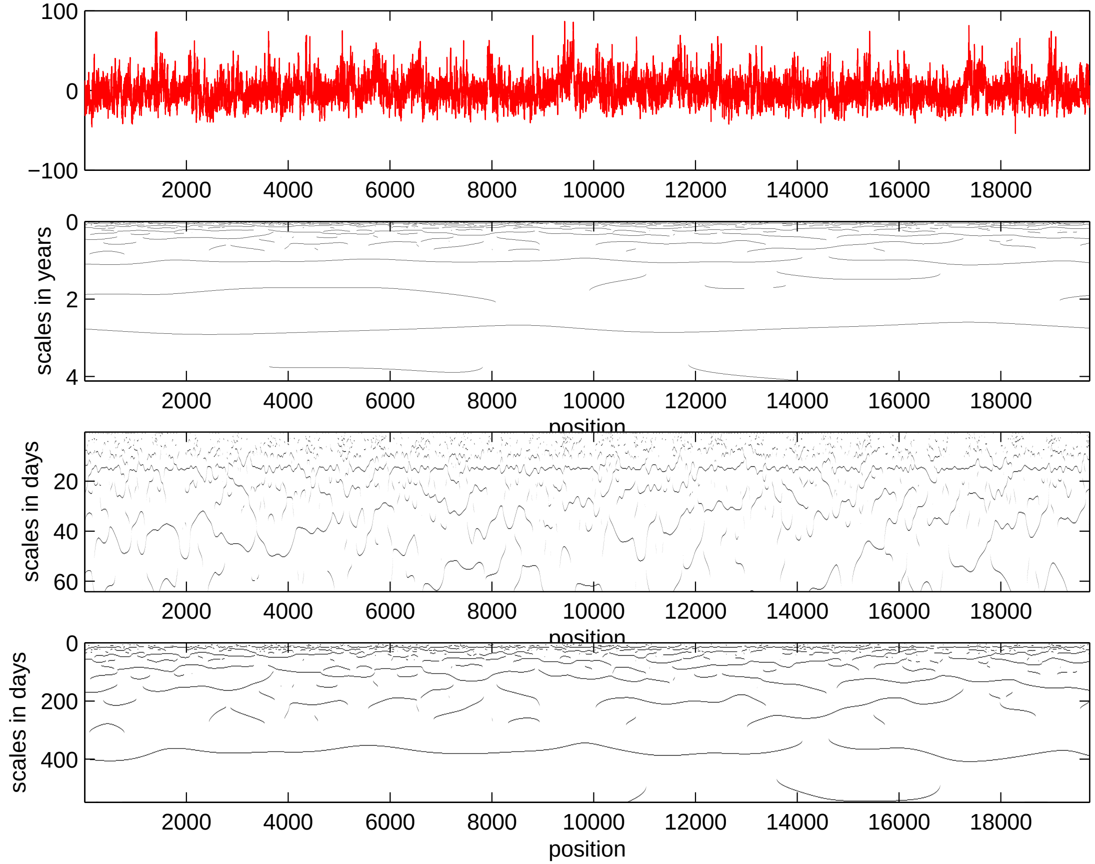
Scales in years
Date
Instantaneous scale
scales in years
Date
Instantaneous Amplitude
Date
Date

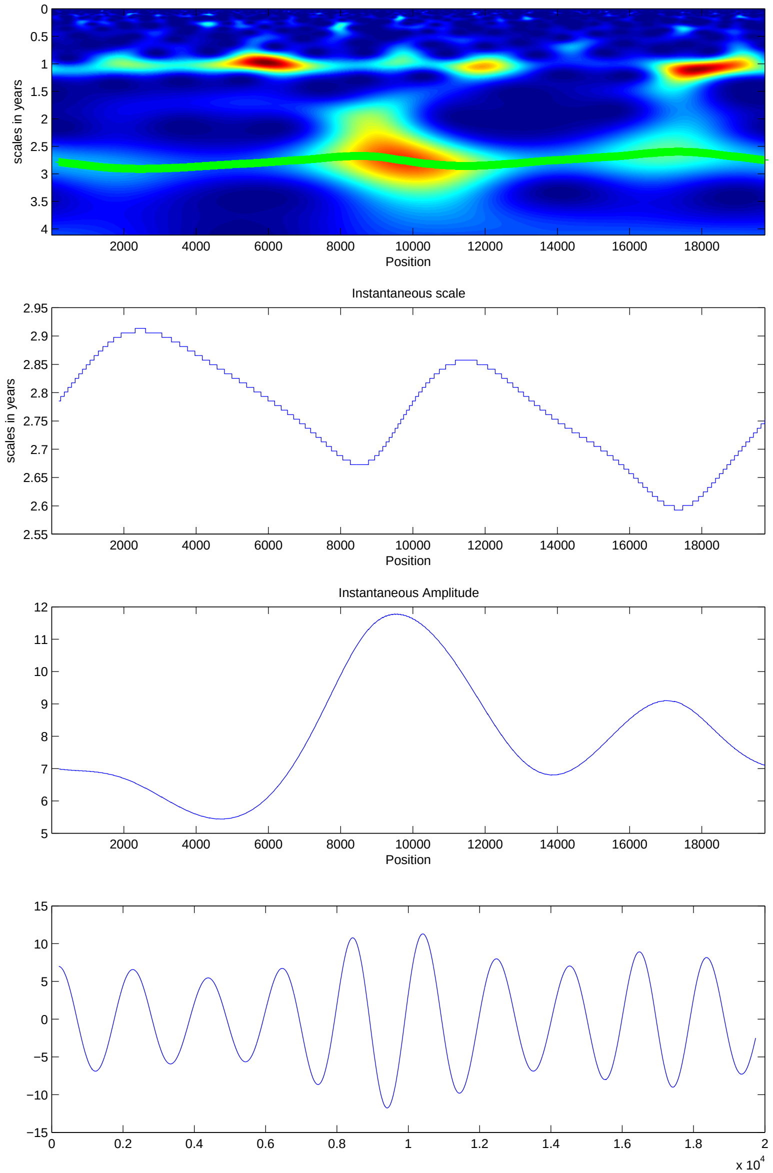
Brest (smoothed and resampled 1 point each 6 hour): 1.01.53-12.01.80

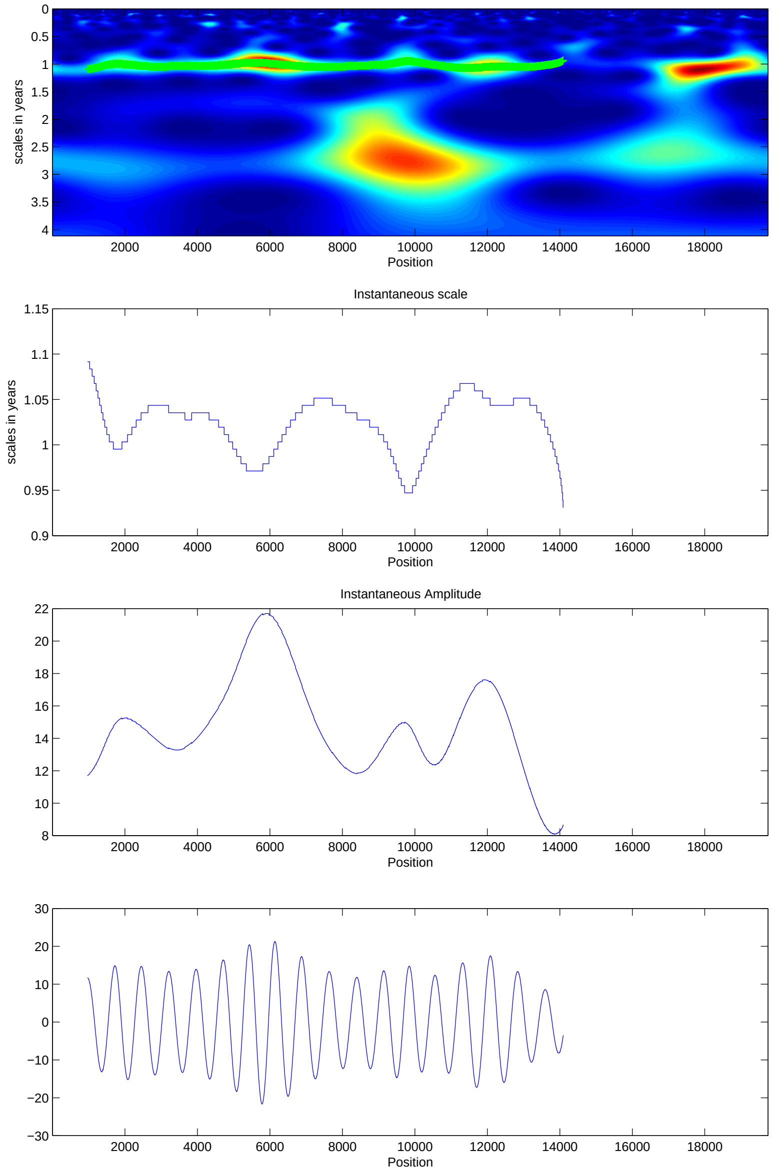
CWT



CWT-Ridges



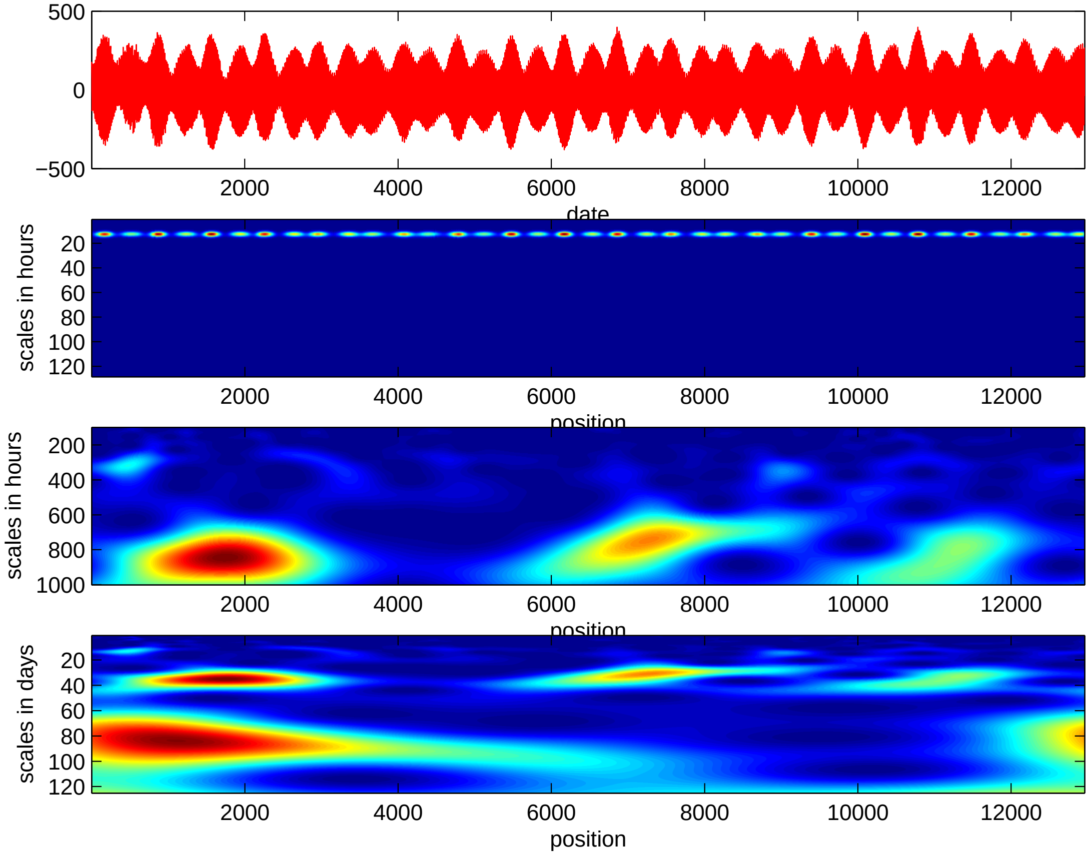

scales in years
Position
Instantaneous scale
scales in years
Position
Instantaneous Amplitude
Position
x 10^4



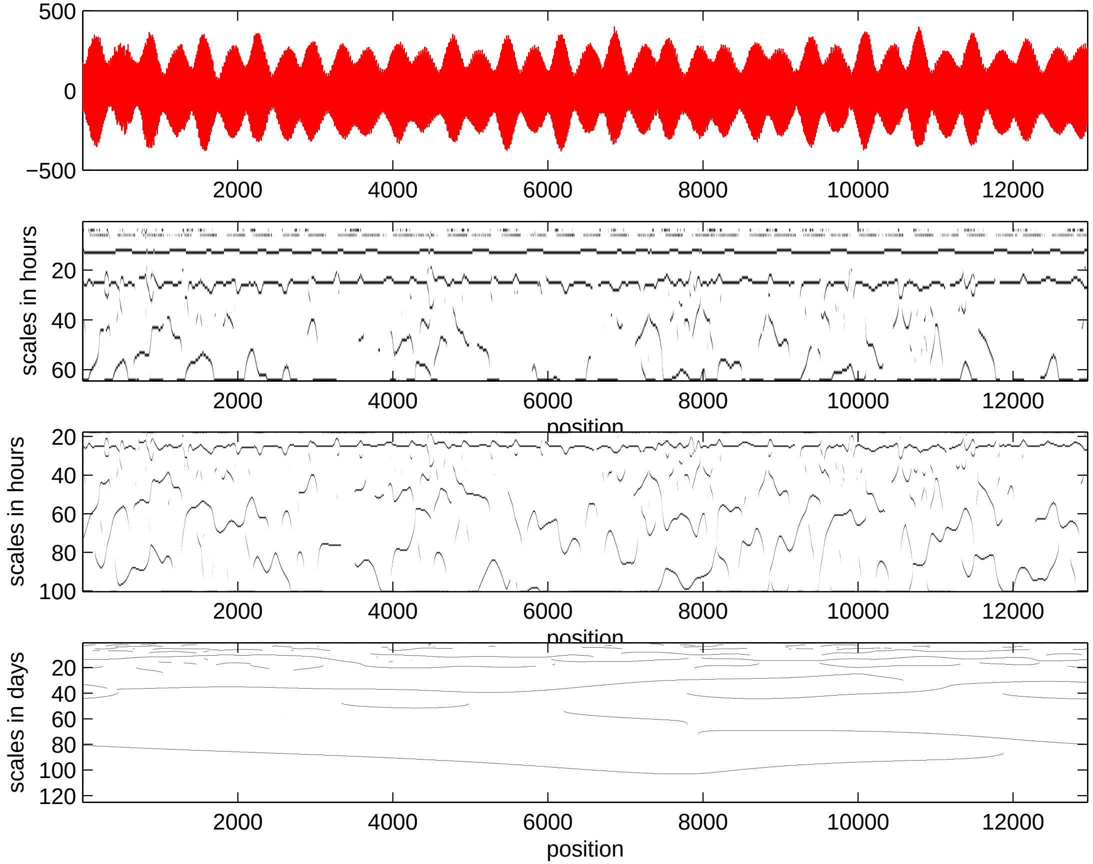
scales in years
Position
Instantaneous scale
scales in years
Position
Instantaneous Amplitude
Position
Position

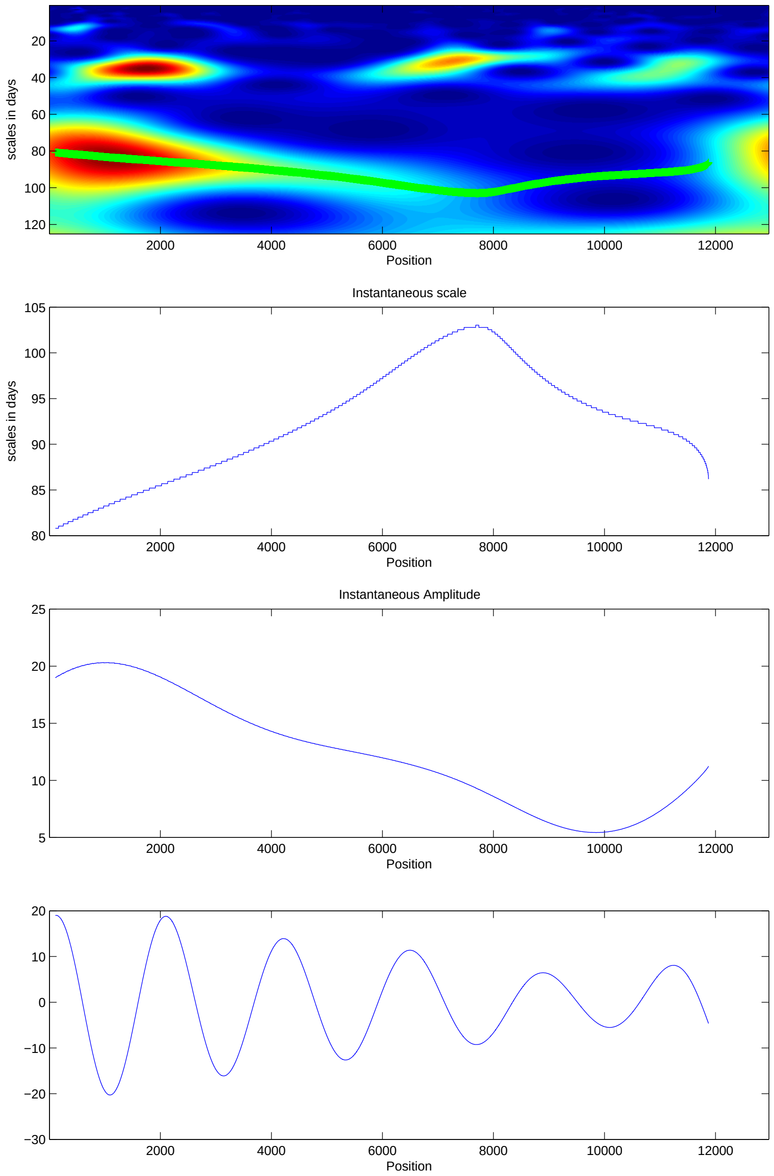
Brest (one sample each hour) 1.06.40-1.05.44

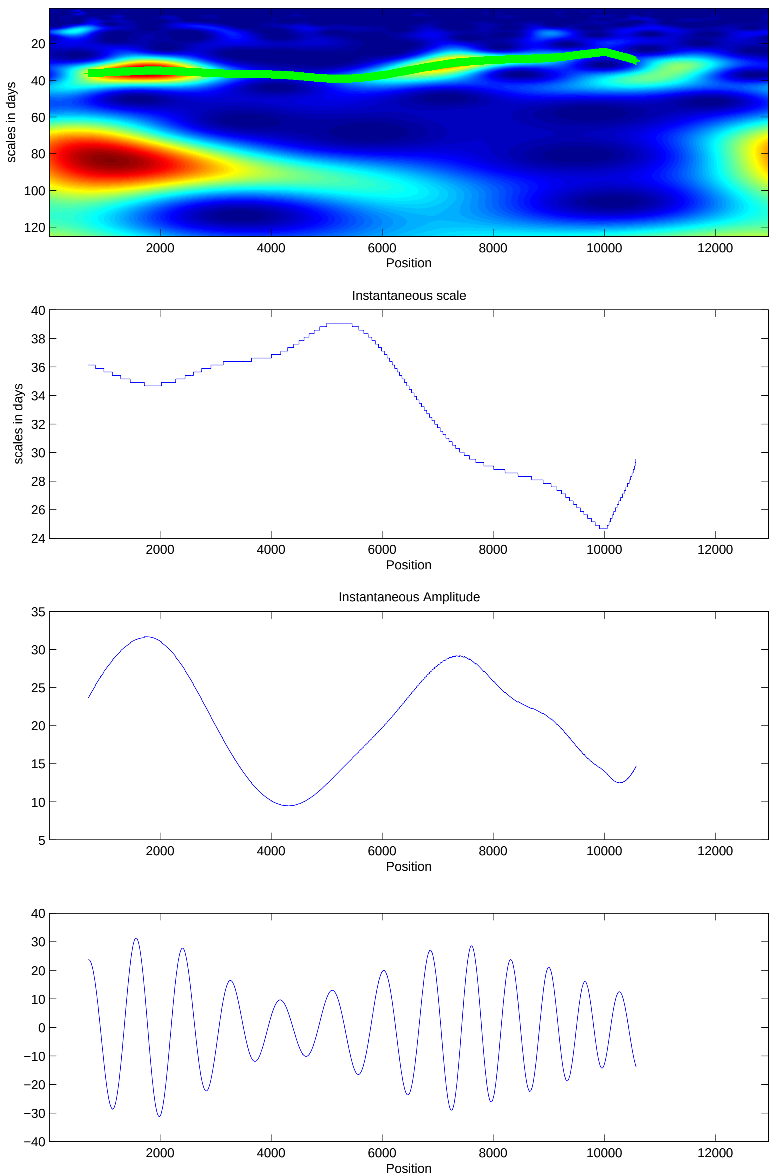
CWT



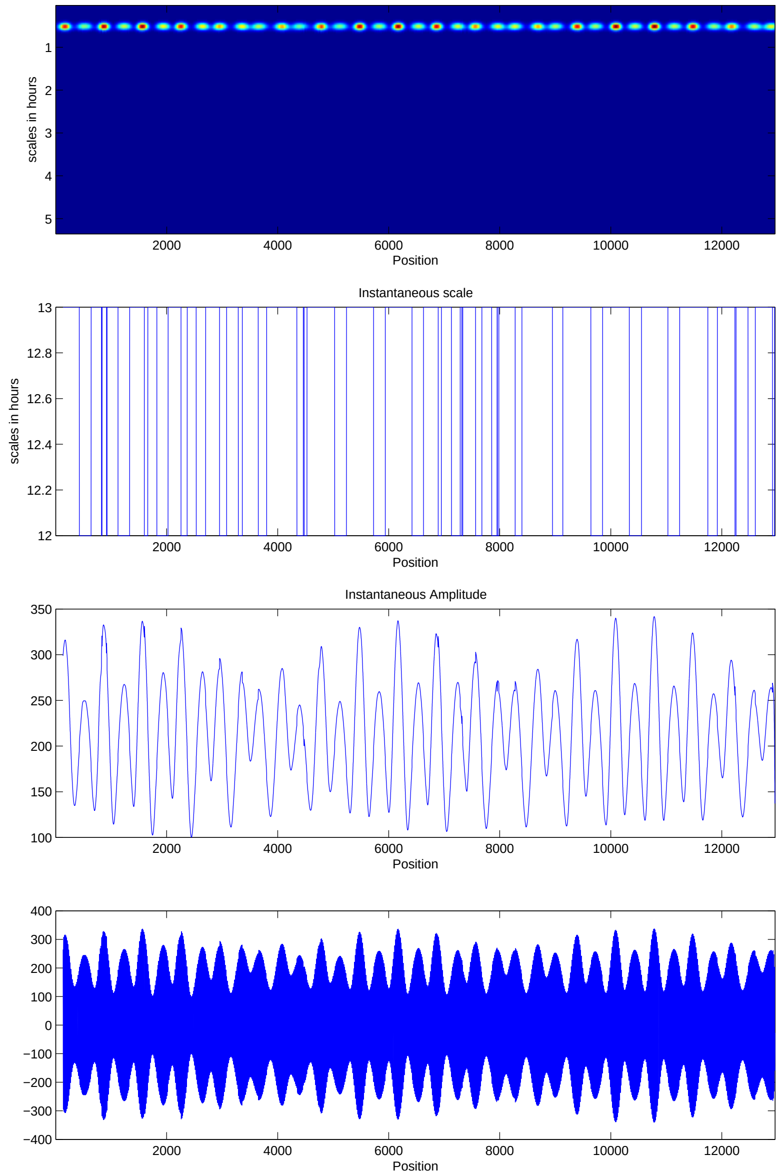
CWT-Ridges




scales in days
Position
Instantaneous scale
scales in days
Position
Instantaneous Amplitude
Position
Position



scales in days
Position
Instantaneous scale
Instantaneous Amplitude


scales in hours
1
2
3
4
5
2000
4000
6000
8000
10000
12000
Position
Instantaneous scale
13
12.8
12.6
12.4
12.2
12
scales in hours
2000
4000
6000
8000
10000
12000
Position
Instantaneous Amplitude
350
300
250
200
150
100
2000
4000
6000
8000
10000
12000
Position
400
300
200
100
0
−100
−200
−300
−400
2000
4000
6000
8000
10000
12000
Position

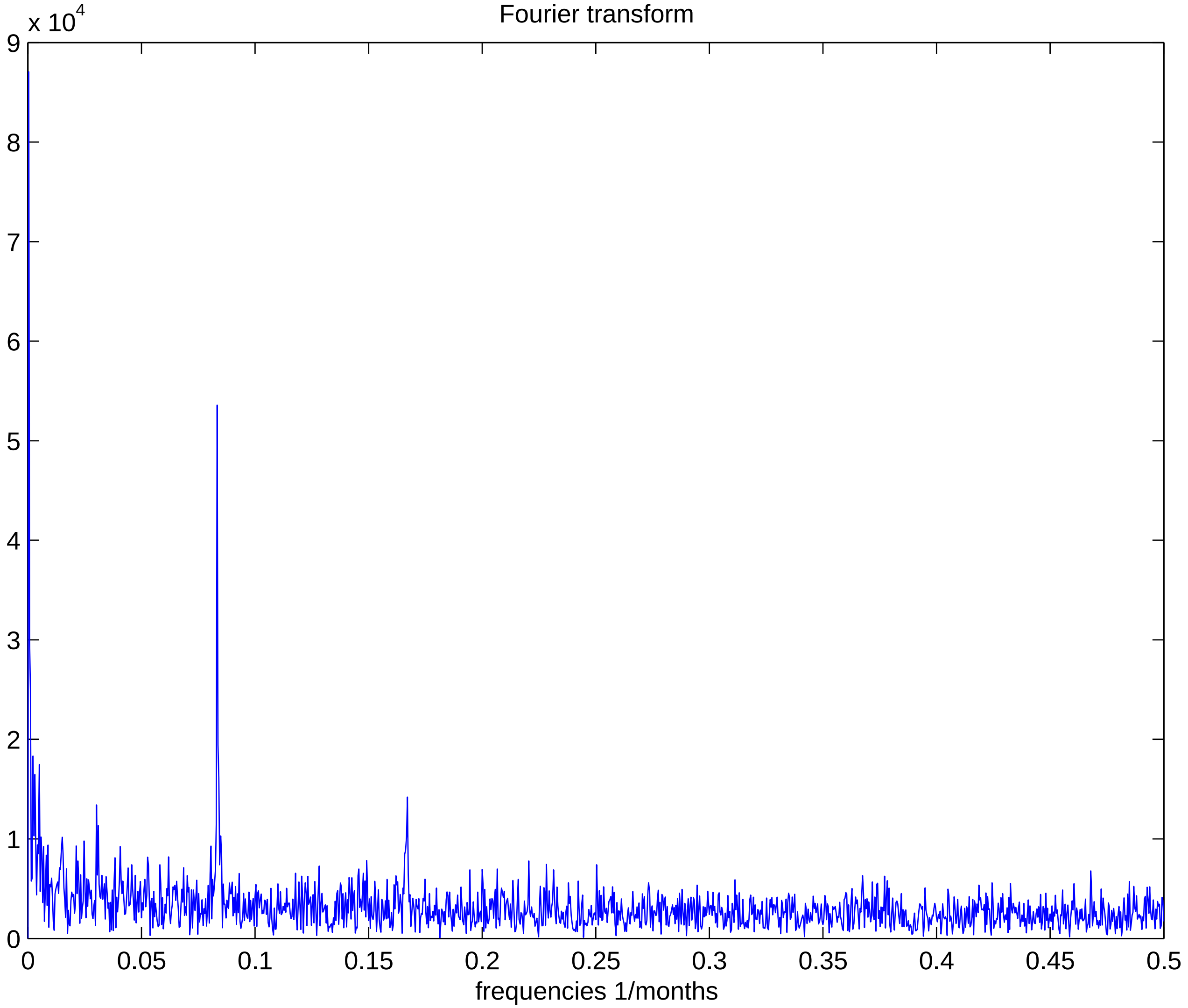
Further reading

- A Wavelet Tour of Signal Processing by Stephane G. Mallat, 1999
- Ten Lectures on Wavelets (Cbms-Nsf Regional Conference Series in Applied Mathematics, No 61) by Ingrid Daubechies, 1992
- Analyse continue par ondelettes by Bruno Torrsani, 1995
- Wavelets: Tools for Science & Technology by Stephane Jaffard, Robert D. Ryan, Yves Meyer, 2001
- A Practical Guide to Wavelet Analysis:
  http://paos.colorado.edu/research/wavelets/
- Wavelet Digest:
  http://www.wavelet.org
- Wavelet Resources:
  http://www.ee.umanitoba.ca/∼ferens/wavelets.html

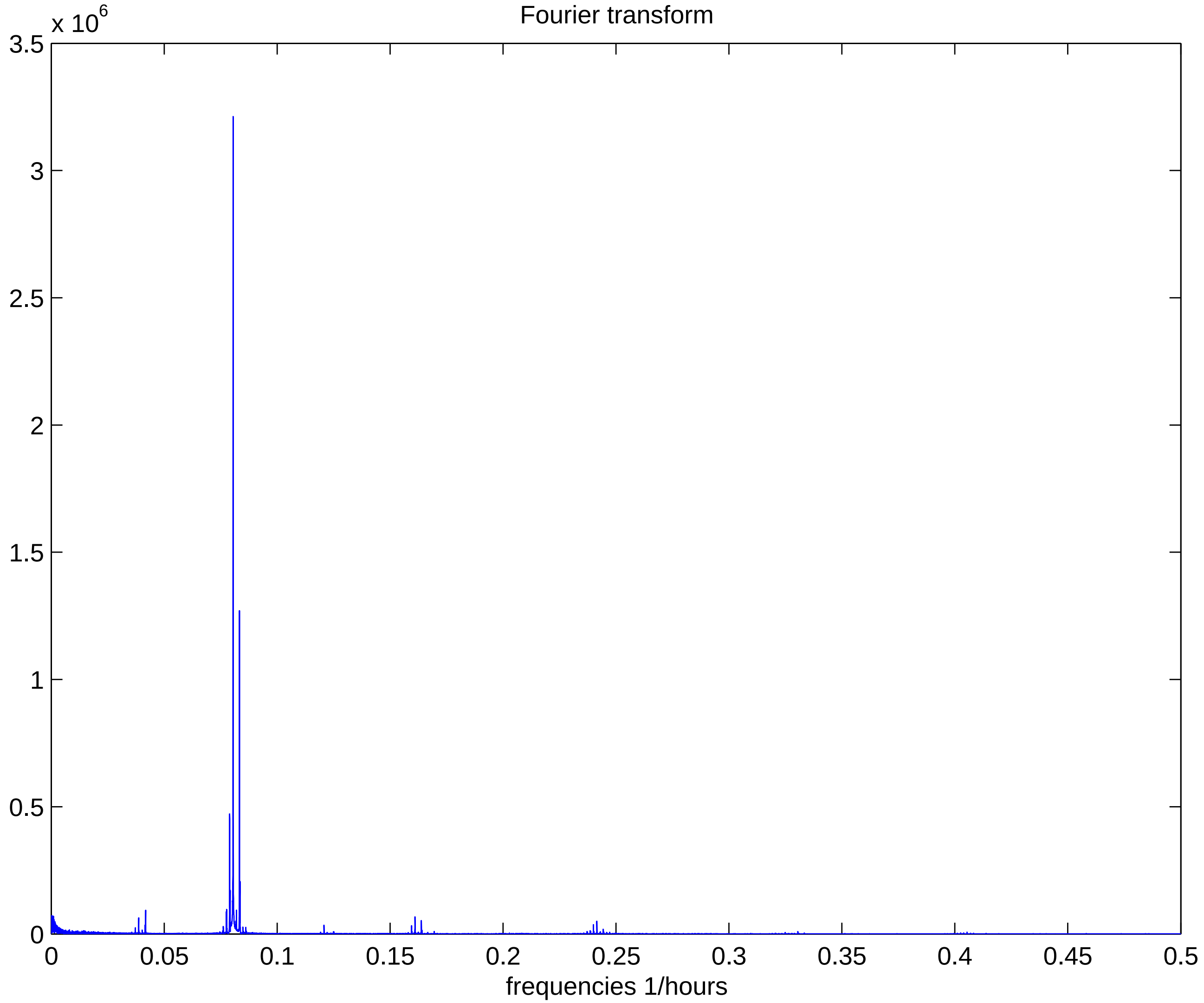
Some free wavelet toolboxes

- YAWT: Yet Another Wavelet Toolbox
  http://www.fyma.ucl.ac.be/projects/yawtb/index.php
- Wavelab: a Wavelet toolbox
  http://www-stat.stanford.edu/∼wavelab
- The Time Frequency Toolbox
  http://crttsn.univ-nantes.fr/∼auger/tftb.html
- LastWave
  http://www.cmap.polytechnique.fr/∼bacry/LastWave/
- ...

FFT Brest

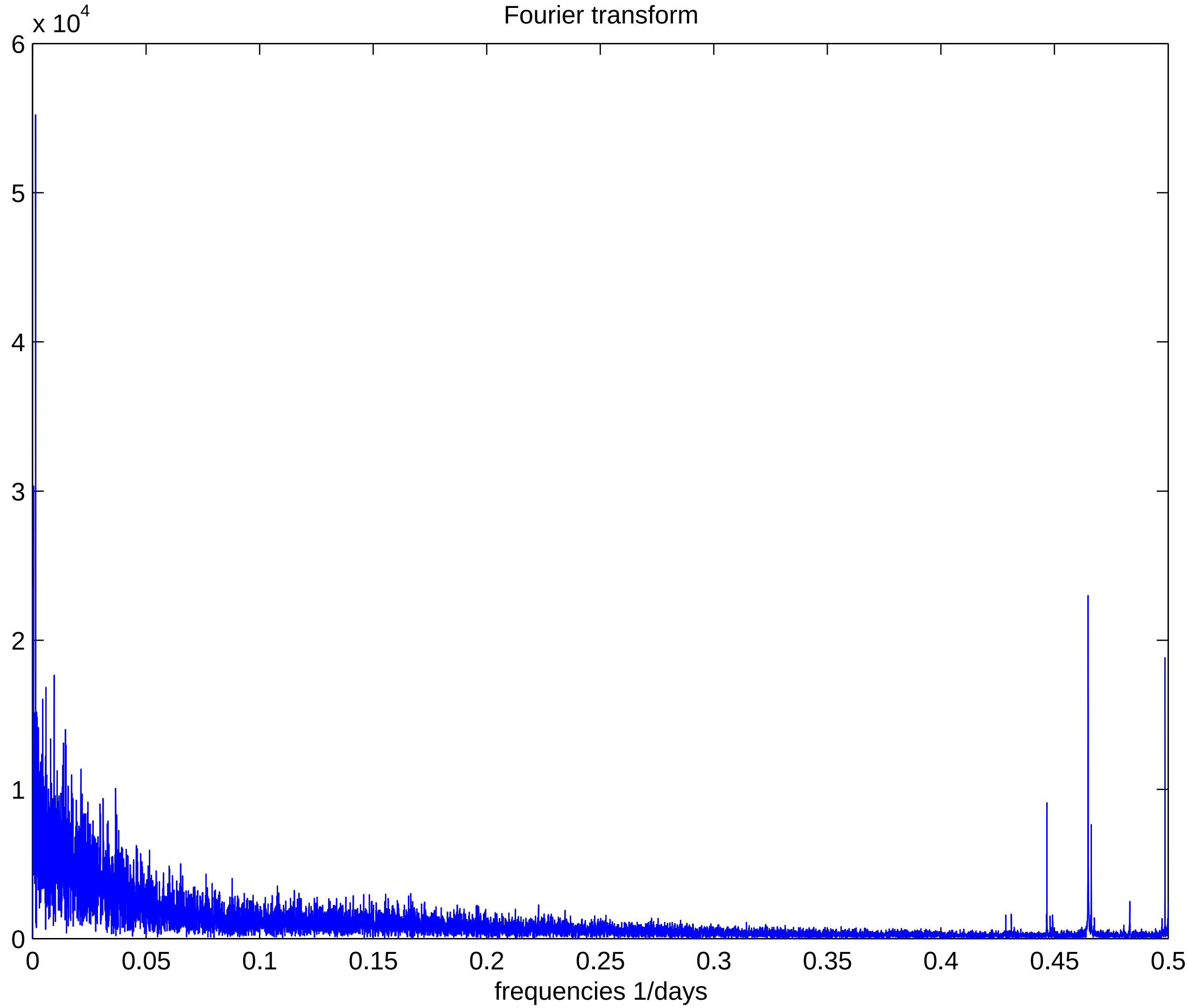

Fourier transform
x 10^4
9
8
7
6
5
4
3
2
1
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
frequencies 1/months

FFT Brest 24



Fourier transform
x 10^6
3.5
3
2.5
2
1.5
1
0.5
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
frequencies 1/hours

FFT Brest 25


Fourier transform
x 10^4
6
5
4
3
2
1
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
frequencies 1/days